A COURSE OF *MATHEMATICS,*

DESIGNED FOR THE USE

OF THE

OFFICERS AND CADETS,

OF THE

ROYAL MILITARY COLLEGE.

By ISAAC DALBY,

Professor of Mathematics in the said College.

VOL. I.

THE SECOND EDITION,

Corrected, with Additions.

LONDON:

PRINTED FOR THE AUTHOR, BY W. GLENDINNING,
NO. 25, HATTON GARDEN.

1807.

CONTENTS.

ARITHMETIC.

1. ARITHMETIC is the science of numbers, or the art of computing by means of the ten numeral digits, or figures: 0 cipher, 1 one, 2 two, 3 three, 4 four, 5 five, 6 six, 7 seven, 8 eight, 9 nine.

All numbers may be denoted by those figures variously combined. And the rule which teaches their different values according to their different places, is called NOTATION, or

NUMERATION.

Let the number 444444444444 be proposed: then the different values of the same figure 4 will be as follows:

Hundreds of thousands of millions	Tens of thousands of millions	Thousands of millions	Hundreds of millions	Tens of millions	Millions	Hundreds of thousands	Tens of thousands	Thousands	Hundreds	Tens	Units
4	4	4	4	4	4	4	4	4	4	4	4

The first figure on the right stands for *four units*, being its simple value; the next for *four tens*, or *forty*, or *ten times* its simple value; the third for *four hundreds*, or a *hundred times*

its simple value; the fourth for *four thousands*, or a *thousand times* its simple value, &c. and the four together or 4444 denote *four thousand four hundred and forty four*.

Hence it appears that the values increase from the right to the left in a decuple proportion, each figure standing for ten times the value of the preceding one.

It is also evident that in reading of numbers there is a constant repetition of *hundreds, tens,* and *units,* at every three figures: thus, the three first on the right denote *four hundred and forty-four;* the next three, *four hundred and forty four thousands;* the next three, *four hundred and forty-four millions;* and the next three, *four hundred and forty-four thousands of millions,* &c.

Therefore in reading of large numbers, if we divide them into periods of six figures each, the first period to the right will be *units, tens, hundreds, and thousands;* the next period will be *millions;* the next *millions of millions, or bi-millions,* or *billions;* the next *tri-millions* or *trillions,* &c. &c.

For example, let 12802410007815104906709 be a proposed number.

12802 410007 815104 906 7 0 9

Then dividing it into periods as above, it will be read thus: *twelve thousand eight hundred and two trillions, four hundred ten thousand and seven billions, eight hundred and fifteen thousand one hundred and four millions, nine hundred and six thousand, seven hundred and nine.*

3. The digits 1, 2, 3, 4, 5, 6, 7, 8, 9, are called *significant figures,* because each has a value by itself, but the *cipher* or *zero* 0 stands for nothing if alone. When annexed however, on the right hand to other figures, or any number, it increases the value ten times: thus 7 denotes only *seven,* but 70 is *seven tens*

or seventy; and 700 *seventy tens or seven hundred;* also 11, signifies only *eleven*, but 110 *eleven tens*, or *one hundred and ten;* 1100 *eleven hundreds*, or *one thousand one hundred;* &c.

And therefore in setting down a proposed number, the places of significant figures must be supplied by ciphers when the former are wanting, as in the following example:

Nine hundred and seventy six	976
Nine hundred and seventy	970
Nine hundred and six	906
Seven thousand nine hundred and six	7906
Seven thousand	7000
Seventeen thousand and six	17006
Ten thousand	10000
One hundred ten thousand and six	110006
One hundred thousand one hundred	100100
One hundred thousand	100000
One million and one	1000001
One million	1000000

OF THE ROMAN NUMERALS OR NOTATION.

4. THE Romans made use of seven capital letters to express numbers,

Namely	I.	V.	X.	L.	C.	D.	M.
Value	1.	5.	10.	50.	100.	500.	1000.

The intermediate [redacted] other numbers, are denoted by two or more of those letters joined or repeated till the sum of the whole make up the proposed number, the characters of the greatest value being set to the left; thus, VI is 6; VII, 7; VIII, 8; and MDCLXVI, 1666. Sometimes a less character is put to the left of a greater, and then it represents their difference as IV, 4; IX, 9; XL, 40; XC, 90; CD, 400. Also IƆ stands for D or 500; and CIƆ for M or 1000. Every C and Ɔ annexed on each side increases the value ten times; thus

CCIƆƆ is 10000. A bar or stroke over a letter increases the value 1000 times, as X̄ is 10000, and C̄ 100000, &c.

This notation is frequently used for the dates, numbering the chapters or sections of books, &c.

SIMPLE ADDITION.

5. Simple Addition consists in finding the sum of two or more numbers of the same denomination. This is done in the following manner:

Place the numbers under each other, so that units are exactly under units, tens under tens, hundreds under hundreds, &c. and draw a line under them. Then add the row of units together, and find how many tens are in the sum.—Set down exactly under the units what remains more than those tens, or when nothing remains, a cipher, and carry one for every ten to the second row.—Next, add up the second row, together with the number carried, then proceed with the sum as before. And in this manner continue the operation till the whole is finished.

Examp. 1. Let the sum of 543 and 216 be required?

543
2[illegible]6
Sum. [illegible]9

[illegible] 2. Required the sum of 578[illegible]1, 4[illegible]0, and 7[illegible]?

57[illegible]
[illegible]
7[illegible]
Sum [illegible]

[illegible] addition I proceed thus:—[illegible] and 0 and 1 make 1[illegible] which is 1 ten and [illegible] over, therefore I put down the [illegible] and carry 1 to the rank of tens; [illegible], [illegible] and [illegible] are 1[illegible] and [illegible] make 19 and 1 I carried make 20, which is 2 tens and 0 over, therefore I put down a cipher and carry 2; again, 7 and 4 make 11 and 8 are 19 and 2 that were carried make 21, which is 1 to put

B

down and 2 to be carried; next, the 2 carried and 7 make 9: lastly, as there is nothing carried to the 5 it becomes the last figure in the sum.

The reason for placing units under units, tens under tens, hundreds under hundreds, &c. and carrying the tens to the left, is manifest from Notation. But because the whole must be equal to the sum of all its parts, if we add together the units in one sum, the tens in another, the hundreds in a third, &c. and add the several sums together, it will prove the addition; and perhaps the reason for carrying the tens will appear more obvious.

The sum of the units........	13
Of the tens....	[illegible]
Of the hundreds	[illegible]
Of the thousands..............	7000
Of the tens of thousands....	50000
Sum	[illegible]103 as before.

6. Another method of proving addition, is to cut off the upper line, then having added all the other lines together, add the upper line to the sum.

Ex. 2. 98764	Proof.	
51238	98764	
72015	51238	
[illegible]	72015	
10392	76958	
8160	10392	
Sum [illegible]	8160	
	[illegible]740	sum without the upper line.
	[illegible]8764	upper line.
Sum	[illegible]	as before.

7. When the numbers to be added are large, and consist of many ranks, divide them into two or more parts, and find the sum of each part separately, then add the several sums together.

Ex. 4.	987654321	Proof.	
	123456789	987654321	
	597763181	123456780	
	790011376	597763184	
	598172867	1703871285	sum
	984799989	790011376	
	624875932	598172867	
	100926793	984799989	
	991876823	2373311212	sum
Sum	5797868075	624875932	
		100926793	
		991876823	
		1720679518	sum

	1708871285	the three sums.
	2373311242	
	1720679518	
Sum	5797868075	as before.

But the usual method of proving Addition is to begin at the upper line and add downwards, in the same manner as it was added upwards, then if the sums agree, we may conclude the work is right.

SIMPLE SUBTRACTION.

8. Simple Subtraction is the operation of taking a less number from a greater, or finding the difference of two proposed numbers: thus, 1 subtracted from 7 leaves 6, which is the difference of 1 and 7; 8 subtracted from 10 leaves 2, the difference of 8 and 10; 22 subtracted from 33 leaves 11 the difference; for 2 units taken from 3 units leaves 1 unit; and 2 tens taken from 3 tens leaves 1 ten; therefore 1 ten and 1 unit, or 11 is the difference. And hence it is evident that in placing numbers for subtraction, units must stand under units, tens under tens, hundreds under hundreds, &c. as in addition.

Ex. 1. From	33
Take	22
Difference or remainder	11

9. The method of proving subtraction is to add the less number and the difference or remainder together, for their sum must evidently be equal to the greater number if the work is right: thus, let the difference of 4356 and 3213 be required.

Ex. 2.	4356
	3213
Difference	1143
Proof	4356 the sum of 3213 and 1143.

10. When the figure to be subtracted is greater than that directly above it, the method of operating is easily derived thus:

Let the difference of 41 and 18 be required:

41
18

differ. 23; here 8 cannot be subtracted from 1, but if 10 is taken from the 40 and added to the 1 the sum is 11, then 8 from 11 and 3 remains; consequently the 1 which stands under the 4 must be subtracted from 3 (or 4 lessened by 1), and the remainder is 2. In like manner proceed with any other number of figures.

Ex. 4. From 823
Take 636

Rem. 187: here 6 from 13 (10 added to 3) and 7 remains; 3 from 11 (10 added to 2 lessened by 1) and 8 remains; 6 from 7 (8 lessened by 1) and 1 remains. But it evidently comes to the same thing if we augment the lower figures by 1 instead of lessening the upper figures; thus 6 from 13 and 7 remains; 4 from 12 and 8 remains; 7 from 8 and 1 remains.

Ex. 5. From 14040
Take 3051

Rem. 10989; here 1 from 10 and 9 remains; 5 from 13 (10 added to 4 lessened by 1) and 8 remains; 0 from 10 lessened by 1 and 9 remains; 3 from 4 lessened by 1 and 0 remains: lastly as there is nothing to subtract from the 1, it becomes the left-hand figure of the remainder

If we augment the lower figures by 1 instead of diminishing the upper ones, the process will be thus: 1 from 10 and 9 remains; 6 from 14 and 8 remains; 1 from 10 and 9 remains; 4 from 4 and 0 remains.

Ex. 6.	From	1000001	*Ex.* 7.	From	81010215
	Take	1101		Take	901016
	Rem.	998900		Rem.	80109199
	Proof	1000001		Proof	81010215

11. Or subtraction may be performed by setting down such figures for the remainder that when added to the less number shall give the greater.

Thus, from 9875
Take 2301

Rem. 7574; here 1 and 4 make 5, therefore 4 is the remainder; 0 and 7 make 7 for the remainder; 3 and 5 make 8, therefore 5 is the remainder; 2 and 7 make 9, therefore 7 is the remainder.

When the lower figure is greater than that directly above, it is evident that the next lower figure must be augmented by 1.

Thus from 10126
Take 1357

Rem. 8769, here 7 and 9 make 16, therefore 9 remain; 6 (or 5 augmented by 1) and 6 make 12, therefore 6 remains; 4 (or 3 augmented by 1) and 7 make 11, therefore 7 remains; 2 (or 1 augmented by 1) and 8 make 10, therefore 8 remains.

SIMPLE MULTIPLICATION.

12. Simple Multiplication consists in finding the sum or amount of a proposed number taken or repeated a given number of times, and may be denominated a compendious method of Addition: for example, suppose 6 is to be taken 3 times:

6
6
6

then the addition gives 18, but by multiplication we say 3 times 6 make 18.

The number to be multiplied is called the *multiplicand*; that by which you multiply, the *multiplier*; and the result is

called the *product*. The multiplicand and multiplier are without distinction called the terms or *factors* of the multiplication, because they *make* the product or number sought: thus 3 times 5 *make* 15.

13. But in the first place it will be necessary to learn perfectly the following Table, which contains the products of every two of the 9 digits.

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

To find the product of two figures in this table, look for one of them in the left-hand column, and for the other at top, then their product will be found where the vertical column from the top intersects the horizontal one from the left. Let 6 and 7 be proposed, then the columns meet at 42; for 6 times 7, or 7 times 6 make 42.

14. The rule for multiplying by a single figure is derived from addition; thus: Let the sum of 3 times 875, or, which is the same thing, the product of 875 by 3, be required?

	875	Multiplicand	875
	875	Multiplier	3
	875	Product	2625
Sum	2625		

To perform the addition; 5, 5, and 5 make 15, or 5 more than 1 ten; 7, 7, and 7 make 21, and 1 make 22, or 2 more than 2 tens; next 8, 8, and 8 make 24, and 2 make 26. But in the multiplication we say 3 times 5 make 15, or 5 more than 1 ten; 3 times 7 make 21, and 1 make 22, or 2 over 2 tens; lastly, 3 times 8 make 24, and 2 (for the 2 tens) make 26. Therefore in multiplication, 1 must be carried to the left for every 10 in the products, and the overplus set down as in addition.

Examp. 2.			*Ex.* 3.	
	Multiply	987600543210	Multiply	123456789
	By	7	By	9
	Product	6913203802470		1111111101

15. When the multiplier consists of one figure with ciphers on the right, multiply by that figure, and annex the ciphers to the right of the product.

Ex. 4.	Multiply	11	
	By	300	
	Product	3300	this is evident from Notation.

When the multiplier consists of several figures.

16. Begin at the right, and multiply by each figure separately and set down the products so that the units of the second line may stand under the tens of the first, the units of the third line under the tens of the second, and so on: then add all the products together for the amount.

Ex. 5. Multiply	231
By	323
	693
	462
	693
Product	74613

The reason for setting down the products by the single figures in this manner will be manifest, if we consider that the whole amount must (in the present example) consist of 3 times 231, 20 times 231, and 300 times 231, when added together:

3 times 231	....	693
20 times 231	....	4620
300 times 231	....	69300
	Sum	74613.

Here if the ciphers are cancelled (as having no value in the addition) the first figure of any line, or product by a single figure, must necessarily fall one place to the left of that above it. And hence the rule for multiplying by several figures is educed.

17. When ciphers are between other figures in the multiplier, neglect them, remembering to set down the lines of products as far to the left as they would be if the ciphers were other figures.

Ex. 6.	*Multiply*	5772
	By	230015
		28860
		13088
		17316
		11544
	Product	1327819740

18. If ciphers are at the right hand of one or both factors, find the product of the other figures, to which annex all the ciphers on the right.

Ex. 7.	Multiply	6300
	By	7000
		441
		252
	Product	44100000

19. When one of the factors is the product of two or more single figures, the other factor may be multiplied by one of the figures, and the product by another, and so on: then the last result will be the answer.

Ex. 8. Let 4615 be multiplied by 72, or 9 times 8.

	4615
	9
	41535
	8
Product	332280

The reason of this operation is obvious; for 9 times any number repeated 8 times, is evidently that number repeated 72 times.

Methods of proving Multiplication.

I.

20. MAKE the multiplicand and multiplier change places; then if the products agree, the work is right.

Examp.	Multiply	6847	Proof.
	By	7806	7806
		41082	6847
		54776	54642
		47929	31224
	Product	53447682	62448
			46836
			53447682

II.

21. Find what is over the exact number of nines in the sum of the digits of each factor, then multiply the excesses together, and find the excess above nines in the digits of this product, which excess ought to be the same as the excess above nines in the digits of the whole product or answer.

Examp.	Multiply	836	—— 8, excess above nines
	By	727	—— 7, excess above nines
		5852	
		1672	
		5852	
	Product	607772	—— 2, excess above nines.

The product of the two excesses 8 and 7 is 56, which gives 2 for the excess above nines, the same as the excess in the whole product or amount.

This method of proof is founded on the following property of the number 9;—*every number, the sum of whose digits is an exact number of nines, is itself an exact number of nines.* This is easily proved as follows. A number containing an exact number of tens must consist of the same number of nines and of units; thus 1 nine and 1 unit make 1 ten, 2 nines and 2 units make 2 tens, 7 nines and 7 units make 7 tens; 60 nines and 60 units make 60 tens, &c.; consequently, if the nines are taken out of the tens in any number, the remainder will be as many units as there are tens in that number; for example, the nines taken from the tens in 670 will leave 67 units, and the nines taken from the 6 tens in 67 will leave 6 units, which, with the 7 units, make 13 the sum of the digits in 670, therefore if nines are cast out of 670, the remainder will be 4 (the digits of 13 added, 1 and 3 make 4); and because 4 wants 5 of 9, it is evident that 675, and the sum of its digits, are each an exact number of nines. And the same method of proof will extend to other numbers.

From hence it follows, that when the nines are cast out of any number, and also out of the sum of its digits, the remainders will be the same.

And in multiplication it is also evident, that when the sum of the digits in one factor is an exact number of nines, the sum of the digits in the product will be an exact number of nines.

In the foregoing example where the excesses above nines in the factors are 8 and 7, the product 607772 is the sum of 836 multiplied by 720, and

[illegible] less by 8 multiplied by 7, and 8 multiplied by 7; the two former parts are exact nines (one of the factors in each being nine) and since the latter part (8 multiplied by 7) is the product of the two excesses in the factors, the truth of the rule is manifest.

This method of proving multiplication by casting out the nines, is probably as ancient as the present system of arithmetic, for we find it in L[illegible] de Borgo's *Summa de Arithmetica*, &c. printed in 1494. But [illegible] a convenient rule, there are circumstances in which it may fail; [illegible] figures should be transposed in the product, or the value of one figure [illegible] and another as much too little, or a 9 be set down instead of 0, or the contrary: in all these cases, the excess above nines will evidently be the same as in the true product.

SIMPLE DIVISION.

22. SIMPLE DIVISION consists in finding how often a less number is contained in, or may be taken from a greater number of the same denomination; and is a compendious method [illegible]. Or it is the method of resolving a given number into a proposed number of equal parts. Thus, if 2 and 10 are the numbers, the former is contained 5 times in the latter; or if 10 be divided into 2 equal parts, each part will be [illegible]

23. The number to be divided is called the *dividend*.—[illegible] by which you divide the *divisor*.— And the number of [illegible] the latter is contained in the former is called the *quotient*.

Dividend.
Divisor 2) 10 (5 *Quotient.*

24. *To perform Division.* Find how often the divisor is contained in [illegible] of the left hand figures of the dividend [illegible] just necessary, which will give the first figure in the quotient. Multiply the divisor by this quotient figure and subtract the product from the aforesaid figures of the dividend, then bring down the next figure of the dividend to the right of the remainder. Find

how many times the divisor is contained in the remainder so increased, for the second figure of the quotient, but if it be 0 times, put a cipher, and bring down another figure; then proceed as before till all the figures are brought down.

Examp. 1. Let 83401190 be divided into 2 equal parts.

```
              Dividend
Divisor 2 ) 83401190 ( 41700595 Quotient
            8
            -
            3                          Proof
            2                        41700595
            --                              2
            14                      ---------
            14                       83401190
            --                      ---------
             011
              10
              --
               19
               18
               --
                10
                10
                --
```

In this example the quotient is half the dividend, therefore if we multiply 41700595 by 2, the product will be 83401190.

25. Hence to prove Division, *multiply the divisor and quotient together, then if the product is the same as the dividend, the work is right.*

26. When there is no remainder after the last subtraction, the quotient will be a whole number, as in the preceding example; but if there be a remainder, place it over the divisor with a line between, on the right of the other figures, and you have the *fractional* part of the quotient.

Ex. 2. Let 101 be divided into 2 equal parts.

```
          2 ) 101 ( 50½ quotient, or the half of 101.
               10
               --
Remainder       1
               --
```

The fraction ½ denotes half, or 1 divided into 2 equal parts, and is the fractional part of the quotient.

Ex. 3. Divide 713391049 into 7 equal parts.

```
7 ) 713391049 ( 101913007 quotient, or the answer.
    7
    13                              Proof.
     7                            101913007
    --                                    7
     63                           ---------
     63                           713391049
     --
       9
       7
       --
       21
       21
       ---
        049
         49
```

```
Ex. 4   9 ) 8257576 ( 917508 4/9 quotient.
            81
            --
             15                     Proof
              9                     917508
             --                          9
              67                    -------
              63                    8257572
              --                          4 remainder
               45                   -------
               45                   8257576
               --
                76
                72
Remainder        4
```

If the integral part of the quotient be multiplied by the divisor 9, and the remainder 4 added to the product, the sum is the dividend, as in the proof.

When the divisor however, is only one figure, it is usual to perform the subtraction mentally and set down the quotient under the dividend: thus,

```
9 ) 8257576
    -------
    917508 4/9 quotient.
```

In this division I proceed thus:— the nines in 82 are 9, and 1 over; the nines in 15 is 1, and 6 over; the nines in 67 are 7, and 4 over; the nines in 45 are 5, the nines in 7 0 times, and 7 over, the nines in 76 are 8, and 4 over.

Ex. 5. Let 67550595 be divided into 211 equal parts.

```
211 ) 67550595 ( 320145 quotient or answer.
      633
      ---
       425
       422
       ---
         305
         211
         ---
          949
          844
          ---
          1055
          1055
```

To find how often the divisor (211) is contained in the numbers of the several steps of the operation, first enquire how many times 2 (the left figure of the divisor) is contained in 6 (the left figure of the dividend); this gives 3 for the first figure in the quotient; next, the 2's in 4 are 2 for the second figure; thirdly, 211 the divisor being greater than 30, a cipher or 0 will be the third figure, fourthly, the 2's in 3 give 1; next, the 2's in 9 give 4; and lastly, the 2's in 10 are 5.

27. But when the dividend is a large number, and the divisor consists of several figures, a table may be formed containing the products of the divisor by the several digits, as in the next example:

Ex. 6. Divide 1447859710478 by 1783.

```
                        ⎧1  give   1783
                        ⎪2 ......  3566
                        ⎪3 ......  5349
                        ⎪4 ......  7132
1783 multiplied by      ⎨5 ......  8915
                        ⎪6 ...... 10698
                        ⎪7 ...... 12481
                        ⎪8 ...... 14264
                        ⎩9 ...... 16047
```

```
1783 ) 1447859710478 ( 812035749 11/1783 quotient.
       14264
        2145
        1783
        ----
         3629                      Proof.
         3566                    812035749
         ----                         1783
          6374                  2436107247
          5349                 6496285992
          -----               5684250243
          10250               812035749
           8915              -------------
          -----              1447859710467
           13354                        11
           12481             -------------
           -----             1447859710478
             8737
             7132
            -----
            16058
            16047
            -----
Remainder      11
```

28. Those who are expert in the practice of division, sometimes omit the products, and set down the remainders only.

Thus, (taking the last example.)

```
1783 ) 1447859740478 ( 812035749 11/1783
        2145
         3629
          6374
           10250
            13354
             8737
              16058
                 11 remainder.
```

And the division is sometimes performed without bringing down the figures of the dividend.

```
Thus, 1783 ) 1447859740478 ( 812035749 11/1783.
                 2142325535 ( 1
                  36603370 ( 1
                     11 86
                       1
```

Where the remainders stand under the corresponding figures of the dividend, as before.

In these contracted methods, the remainders are obtained by performing the subtraction while you multiply. Thus to find 214 the first remainder: 8 times 3 make 24, and 4 make 28, therefore 4 is the right-hand figure of the remainder; next 8 times 8 make 64, and 2 (the tens carried) make 66, and 1 make 67, consequently 1 is the next figure; again, 8 times 7 are 56, and 6 (the tens carried) make 62, and 2 make 64, therefore 2 is the other figure of the remainder. And in the same manner the other remainders are found.

29. When the divisor is a number with ciphers on the right, cut them off, and also the like number of figures from the right of the dividend, then divide the remainder of the dividend by that of the divisor in the usual manner, and bring down the figures cut off from the dividend to the right of what remains after this division, if any thing, for the whole remainder; otherwise the figures cut off will be the true remainder.

Ex. 7. Divide 215135 by 2500.

```
25,00 ) 2151,35 ( [illegible] quotient.
        225
         201
         200
    Rem. [illegible]
```

8. Divide 245032 by 2500.

25,00) 2450,32 ($98\frac{32}{2500}$ quotient
225
200
200
Rem. 32

9. Divide 715640 by 6000.

6,000) 715,640
$119\frac{1640}{6000}$ quotient, the remainder being 1640.

10. Divide 6421 by 10.

1,0) 642,1
$642\frac{1}{10}$ quotient, the remainder being 1.

30. When the divisor is the product of two or more small figures, divide by one of those figures, and the quotient by another, and so on.

Ex. 11. Divide 332280 by 72, or 9 times 8. (See *Example* 7, in Multiplication.)

9) 332280
8) 36920
4615 quotient.

The method of finding the true quotient when there are remainders, belongs to Vulgar Fractions, to which we refer for an example.

Since the product of the divisor and quotient (without the fractional part, should there be any) gives the dividend lessened by the remainder, it is evident that division may be proved by casting out the 9's exactly in the same manner as multiplication.

OF VULGAR FRACTIONS.

31. THE operations by common arithmetic extend to integers only, unity or one being the least number in the computations. When parts, or quantities less than 1, are the subject of consideration, it is called *Fractional Arithmetic*. A *fraction* there-

force is properly an expression for part of an *unit* or the integer 1. This integer [illegible] may represent a *whole* of any kind, and the parts into which it is broken, or supposed to be divided, are *fractions* of that *whole*.

Thus if 1 pound is the integer, and we divide it into 20 equal parts, 1 of [illegible] parts, or a shilling, will be represented by the fraction $\frac{1}{20}$ *(one twentieth)*; and 7 shillings by the fraction $\frac{7}{20}$ *(seven twentieths)*. If a foot in length [illegible] the integer, the expression for 1 inch will be $\frac{1}{12}$ *(one twelfth)*; but if we take a yard for the integer, 1 inch will be denoted by $\frac{1}{36}$ *(one thirty-sixth)*, because 36 inches make a yard.

32. A fraction also arises from division in whole numbers when there is a remainder; or when the divisor is greater than the dividend: in the former case it is part of the quotient (see examples 2, 4, &c. in simple division), and in the latter, the quotient itself.

Thus if 5 be divided by 2 the quotient is $2\frac{1}{2}$. And 3 divided by 4 gives $\frac{3}{4}$ for the quotient. Here the fractions are $\frac{1}{2}$ and $\frac{3}{4}$: the former ($\frac{1}{2}$) being [illegible], or 1 divided by 2; and the latter ($\frac{3}{4}$) three-fourths, or 3 divided by 4, or [illegible] of 3.

33. The lower figure of a fraction (denoting the number of parts into which the integer or 1 is supposed to be divided) is called the *denominator*; and the upper figure (which shews the number of [illegible] expressed by the fraction) the *numerator*; thus 4 is the denominator, and 3 the numerator of the fraction $\frac{3}{4}$. Also both are generally named the terms of the fraction.

34. Fractions are either *proper*, *improper*, *simple*, or *compound*.

A *proper fraction* is when the numerator is less than the denominator, as $\frac{1}{2}$, or $\frac{2}{3}$, or $\frac{7}{12}$, &c. and therefore it is always less than 1.

An *improper fraction* has the numerator equal to, or greater than the denominator, and consequently its value must be equal to, or greater than 1. Thus $\frac{4}{4}$ is an improper fraction, because

it denotes 1 or a whole; for four fourths make a whole. $\frac{7}{4}$ is also an improper fraction, it being the same as 7 quarters or 1 and $\frac{3}{4}$.

A *simple fraction* is any fraction having only one numerator, and one denominator, as $\frac{2}{3}$, or $\frac{3}{11}$.

A *compound fraction* is the fraction of a fraction, or several single or simple fractions connected with the word *of* between them: thus $\frac{2}{3}$ *of* $\frac{3}{4}$, and $\frac{5}{7}$ *of* $\frac{3}{8}$ *of* $\frac{1}{2}$, are compound fractions. Also if 1 pound be the integer, the compound fraction $\frac{1}{2}$ *of* $\frac{1}{20}$ will denote sixpence, it being the $\frac{1}{2}$ of 1 shilling or of $\frac{1}{20}$ of a pound.

A *mixt number* is composed of an integer and a fraction, as $5\frac{1}{3}$, $20\frac{5}{11}$, &c.

A whole number may be expressed like a fraction by placing 1 under it as a denominator: thus $\frac{12}{1}$ denotes 12 units, or 12.

A *prime number* is that which can only be measured by 1, or unity: thus 2, 3, 5, 7, 11, &c. are prime numbers.

A *composite number* is that which can be measured by some number greater than 1: or it is the product of two or more numbers: thus 4, 6, 8, &c. are composite numbers.

---

36. THE familiar use of the characters $=$, $+$, $-$, $\div$, will greatly abbreviate the operations in vulgar fractions.

$=$ signifies *equal to*:

As 12 pence $=$ 1 shilling,
12 inches $=$ 1 foot,
3 feet $=$ 1 yard,
$\frac{1}{2}$ an hour $=$ 30 minutes, &c.

$+$ (*plus*) the character for *addition*:

Thus $2 + 3 = 5$, 2 added to 3 are equal to 5.
$4 + 6 = 7 + 3$, 4 added to 6 are equal to 7 and 3.

$-$ (*minus*) signifies *subtraction*:

As $5 - 3 = 2$ 3 subtracted from 5 is equal to (or leaves) 2.
$4 - 3 = 2 - 1$, the difference of 4 and 3 is equal to that of 2 and 1.

$\times$ the character for *multiplication*:

$2 \times 3 = 6$, 2 multiplied by 3 is equal to (or produces) 6.
$2 \times 3 \times 4 = 24$, the continual product of 2, 3, and 4, is equal to 24.
$\frac{7 \times 3}{5 \times 4} = \frac{21}{20}$, the fraction $\frac{7 \times 3}{5 \times 4}$ is equal to the fraction $\frac{21}{20}$.

$\div$ the character sometimes used to signify *division*.

As $24 \div 4 = 6$, 24 divided by 4 is equal to (or produces) 6.
$5 \div 2 = 2\frac{1}{2}$, 5 divided by 2 is equal to $2\frac{1}{2}$.
$3 \div 4 = \frac{3}{4}$, 3 divided by 4 is equal to $\frac{3}{4}$.

37. But the proper method of abbreviating division is to set down the quotient in the form of a fraction by placing the divisor under the dividend; thus, 3 divided by 4 gives $\frac{3}{4}$ for the quotient; 5 divided by 2 gives the quotient $\frac{5}{2}$; and 1 divided by 4 produces $\frac{1}{4}$, or a quarter. In general, every fraction should be considered as the quotient arising from the division of the numerator by the denominator.

REDUCTION OF VULGAR FRACTIONS.

38. REDUCTION of Vulgar Fractions principally consists in changing them to a more commodious form for the operations of addition, subtraction, &c.

CASE 1. *To abbreviate or reduce fractions to their lowest terms.*

39. IF the terms of a fraction are multiplied or divided by any

number, its value will evidently remain the same as before; thus, the numerator and denominator of $\frac{3}{6}$ multiplied by 4 produces the fraction $\frac{12}{24}$, or divided by 3 gives $\frac{1}{2}$ (or half), the same as $\frac{12}{24}$ or $\frac{3}{6}$. Therefore to reduce a fraction to its lowest terms, divide the terms of the fraction by any number that will leave no remainder, and the quotients again by the same, or any other number, and so on, till 1 is the greatest divisor; then the fraction will be in its lowest terms.

Ex. 1. Reduce $\frac{1408}{1664}$ to its lowest terms.

This fraction may be reduced by a continual division by 2, thus:

$$2)\ \frac{1408}{1664} = \frac{704}{832} = \frac{352}{416} = \frac{176}{208} = \frac{88}{104} = \frac{44}{52} = \frac{22}{26} = \frac{11}{13} \text{ the lowest terms.}$$

Therefore $\frac{1408}{1664}$ is equal to $\frac{11}{13}$.

When 2 fails as a divisor, try 3, 5, or 7, because if a number is divisible by any digit, (1 excepted) it must be divisible by either 2, 3, 5, or 7.

Ex. 2. Reduce $\frac{1470}{2205}$ to its lowest terms.

$$5)\ \frac{1470}{2205} \overset{3)}{=} \frac{294}{441} \overset{7)}{=} \frac{98}{147} \overset{7)}{=} \frac{14}{21} = \frac{2}{3}.$$ *Ans.* where 5, 3, 7, 7, are the divisors.

Ex. 3. Reduce $\frac{36300}{231000}$ to its lowest terms.

$$\frac{36300}{231000} = \frac{363}{2310} = \frac{121}{770} = \frac{11}{70}.$$ *Ans.* where the divisors are 100, 3, and 11.

40. If the numerator and denominator are large numbers, find their greatest divisor, or common measure, by the following rule: *Divide the greater by the less, and the last divisor by the last remainder, and so on, till nothing remains; then the last divisor is the greatest common measure required.*

If 1 remains for the last divisor, the numerator and denominator (having 1 for their greatest common measure) are said

to be prime to each other; and the fraction is already in its lowest terms.

Ex. 4. Reduce $\frac{7631}{26415}$ to its lowest terms.

$$
\begin{array}{l}
7631\,)\,26415\,(\,3 \\
\quad\;\; \underline{22893} \\
\quad\;\; 3522\,)\,7631\,(\,2 \\
\qquad\qquad\;\; \underline{7044} \\
\qquad\qquad\;\; 587\,)\,3522\,(\,6 \\
\qquad\qquad\qquad\quad\;\; \underline{3522}
\end{array}
$$

Therefore the last divisor 587 is the greatest number that will divide 7631 and 26415 without leaving any remainder.

$$587\Big)\frac{7631}{26415}\Big(\frac{13}{45} \text{ *the fraction in its lowest terms.*}$$

In like manner the greatest divisor or common measure of three or more numbers may be found. For having found the greatest common measure of two of them, as above, find the greatest divisor of that common measure and another of the numbers, and so on. Thus 15 is the greatest common measure of 1995, 840, and 600.

The foregoing rule for finding the greatest common divisor of two numbers is founded on the following axiom; *if a number measures another number, and also a part of that number, it will measure the remaining part.* Thus 5 measures 40 (or 5 is contained in 40 an exact number of times), and it also measures 25 (a part of 40), therefore it measures 15 the other part. That the operation brings out the greatest divisor may be shewn from the 4th example, thus:—The denominator 26415 is equal to the numerator $7631 \times 3 + 3522$ (by the method of proving common division): now if there is a greater divisor than 587 which measures 7631, and $7631 \times 3 + 3522$, it must (by the preceding axiom) measure 3522. And for the like reason, if it measures 3522, it must measure 3522×2. And if it measures 7631 and 3522×2, it must (by the same axiom) measure their difference, or $7631 - 3522 \times 2$, or 587, viz. the greater measures the less, which is absurd.

CASE 2. *To reduce an improper fraction to its equivalent whole or mixt number.*

41. THIS is evidently nothing more than common division. Therefore divide the numerator by the denominator, and the quotient will be the answer.

Ex. 1. Reduce $\frac{957}{43}$ to a whole, or mixt number.

```
43 ) 957 ( 22 11/43 Answer.
     86
     ---
      97
      86
      --
      11
```

2. Reduce $\frac{54800}{2740}$ to its whole, or mixt number.

```
2740 ) 54800 ( 20 Answer.
       5480
       ----
          0
```

3. Reduce $\frac{7200}{1000}$ to its whole, or mixt number.

```
1000 ) 7200 ( 7 1/5 Answer.
       7000
       ----
        200
```

CASE 3. *To reduce a mixt number to its equivalent improper fraction.*

42. THIS operation is the reverse of the former; therefore multiply the whole number by the denominator of the fraction, and add the numerator to the product, then place the sum over the denominator for the fraction required.

Example. Reduce $22\frac{11}{43}$ to an improper fraction.

```
  22
  43
  --
  66
 88
 ---
 946
  11
 ---
 957        957/43 Answer.
```

Hence to reduce a whole number to an improper fraction having a given denominator:—multiply the said number by the proposed denominator, and make the product the numerator of the required fraction.

Example. Let 13 be reduced to a fraction whose denominator is 7.

$13 \times 7 = 91$ the numerator. *Answer* $\frac{91}{7}$.

For $\frac{91}{7} = 13$ by the preceding article.

CASE 4. *To reduce a compound fraction to an equivalent simple one.*

43. MULTIPLY all the numerators together for the numerator, and all the denominators together for the denominator of the fraction required.

If part of the compound fraction be a mixt, or a whole number, reduce the former to an improper fraction, and make the latter a fraction by placing 1 under it as a denominator.

Ex. 1. Reduce $\frac{1}{2}$ *of* $\frac{3}{4}$ to a simple fraction.

$$\frac{1}{2} \times \frac{3}{4}, \text{ or } \frac{1 \times 3}{2 \times 4} = \frac{3}{8} \textit{ the fraction required.}$$

Ex. 2. Reduce $\frac{2}{3}$ *of* $\frac{5}{7}$ *of* $3\frac{1}{3}$ *of* 4 to a simple fraction.

First $3\frac{1}{3} = \frac{10}{3}$; and $4 = \frac{4}{1}$;

Then $\frac{2}{3} \times \frac{5}{7} \times \frac{10}{3} \times \frac{4}{1} = \frac{400}{63}$ *answer.*

44. When a like number of like factors are found in the numerator and denominator, cancel them in both.

Ex. 3. Reduce $\frac{2}{3}$ *of* $\frac{1}{2}$ *of* $\frac{3}{5}$ *of* $\frac{5}{7}$ *of* $\frac{3}{4}$ to a simple fraction.

$\frac{2 \times 1 \times 3 \times 5 \times 3}{3 \times 2 \times 5 \times 7 \times 4}$ here cancelling 2, 3, and 5, in both numerator and denominator, the fraction becomes $\frac{1 \times 3}{7 \times 4} = \frac{3}{28}$ the *answer*. This is reducing the fraction to lower terms by means of the divisors 2, 3, and 5. (39)

The rule for reducing compound fractions may be derived as follows:— Suppose a shilling to be the integer; then because 48 farthings make 1 shilling, the simple fraction denoting 3 farthings is $\frac{3}{48}$, and the compound fraction will be $\frac{3}{4}$ *of* $\frac{1}{12}$ $\left(\text{or } \frac{3}{4} \text{ of a penny}\right)$, and the respective products of the numerators, and the denominators give $\frac{3 \times 1}{4 \times 12}$ or $\frac{3}{48}$ the simple fraction.

Or more generally thus: let $\frac{3}{4}$ *of* $\frac{5}{7}$ be the compound fraction. Then because $\frac{4 \times 5}{4 \times 7}$ is $= \frac{5}{7}$, the fraction $\frac{5}{4 \times 7}$ will be $\frac{1}{4}$ *of* $\frac{5}{7}$, consequently $\frac{3 \times 5}{4 \times 7}$ will be 3 times $\frac{5}{4 \times 7}$, or $\frac{3}{4}$ *of* $\frac{5}{7}$. And in the same manner we may proceed with any number of fractions, first reducing two of them to a simple fraction, and then taking that and a third, and so on.

Hence it appears that the word *of* in a compound fraction signifies *multiplication.*

Case 5. *To reduce fractions of different denominators to equivalent fractions having a common denominator.*

45. The general rule for this purpose may be derived thus. Let the fractions $\frac{2}{3}$, $\frac{5}{7}$, and $\frac{1}{11}$ be proposed.

Multiply the terms of the fraction $\frac{2}{3}$ by the denominator 7, and we have $\frac{2 \times 7}{3 \times 7} = \frac{2}{3}$. (39)

And the terms of the fraction $\frac{5}{7}$ multiplied by the denominator 3 give $\frac{3 \times 5}{3 \times 7} = \frac{5}{7}$.

Therefore the fractions $\frac{2 \times 7}{3 \times 7}$ and $\frac{3 \times 5}{3 \times 7}$ $\left(\text{or } \frac{14}{21} \text{ and } \frac{15}{21}\right)$ having the common denominator 21, are respectively equal to the fractions $\frac{2}{3}$ and $\frac{5}{7}$.

Next, taking $\frac{15}{21}$ and $\frac{1}{11}$, and multiplying the terms of the former fraction by 11, and those of the latter by 21, we get $\frac{15 \times 11}{21 \times 11} = \frac{15}{21}$, and $\frac{1 \times 21}{11 \times 21} = \frac{1}{11}$.

Therefore the fractions $\frac{15 \times 11}{21 \times 11}$ and $\frac{1 \times 21}{11 \times 21}$ having the common denominator 21×11, are respectively equal to $\frac{15}{21}$ and $\frac{1}{11}$, or $\frac{5}{7}$ and $\frac{1}{11}$.

And if the terms of the fraction $\frac{2\times7}{3\times7}$ (or $\frac{2}{3}$) are multiplied by 11, we have $\frac{2\times7\times11}{3\times7\times11}=\frac{2\times7}{3\times7}$ (or $\frac{2}{3}$).

Consequently the three fractions $\frac{2\times7\times11}{3\times7\times11}$, $\frac{[illegible]}{3\times7\times11}$, $\frac{1\times3\times7}{3\times7\times11}$, having the common denominator $3\times7\times11$, are equal to $\frac{2}{3}$, [illegible], $\frac{1}{11}$ respectively. And the same method may be extended to any number of fractions.

Hence it appears that the new numerators are found by multiplying each numerator into all the denominators except its own, and that the common denominator is the continued product of all the denominators.

Ex. 2. Reduce $\frac{6}{7}$, $\frac{5}{9}$, and $\frac{2}{3}$ to equivalent fractions having a common denominator.

$$\left.\begin{array}{l} 6\times9\times3=162 \\ 5\times7\times3=105 \\ 2\times9\times7=126 \end{array}\right\}\text{ the numerators.}$$

$$7\times9\times3=189 \text{ the common denominator}$$

and the fractions are $\frac{162}{189}$, $\frac{105}{189}$, $\frac{126}{189}$, or $\frac{54}{63}$, $\frac{35}{63}$, $\frac{42}{63}$, when abbreviated.

When any factors in the new numerators and common denominator have a common measure or divisor, resolve them into other factors, then (30) reject the like number of like factors in the numerators and denominator, and the fractions will be reduced to the lowest terms which admit of a common denominator.

Ex. 3. Let $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{9}$, be reduced to a common denominator.

The fractions with a common denominator are $\frac{6\times9}{4\times6\times9}$, $\frac{4\times9}{4\times6\times9}$, and $\frac{4\times6}{4\times6\times9}$: now 2, and 3, are the respective divisors of 4 and 6, and 6 and 9; therefore if 6 in the first and third fractions, and 6, 4 and 9 in

the second, are resolved into the factors 2 and 3, the fractions will be $\frac{2\times3\times9}{4\times2\times3\times9}$, $\frac{2\times3\times2\times3}{4\times2\times3\times9}$, and $\frac{4\times2\times3}{4\times2\times3\times9}$, and rejecting 2×3 in the numerators and denominators, we have $\frac{9}{4\times9}$, $\frac{2\times3}{4\times9}$, and $\frac{4}{4\times9}$, or $\frac{9}{36}$, $\frac{6}{36}$, and $\frac{4}{36}$; where the common denominator 36 is the least common multiple or number divisible by 4, 6, and 9. And in the same manner the least common multiple of other proposed numbers may be found, first making them the denominators of fractions having 1 for each numerator.

46. But the least common multiple is readily found by the following rule. (See *art.* 212. vol. 2.)

Write down the proposed numbers in a line, and divide by the prime number 2 as long as it will divide two or more of them without a remainder, and set down the quotients together with the undivided numbers in a line below.——Divide this second line by 2, and also the third line, &c. in the same manner, if they will divide. This done, proceed with 3 the next prime number, and so on to 5, or 7, &c. till there are no two numbers that can be thus divided: Then the continued product of the divisors, the last quotients, and the undivided numbers, is the multiple sought.

Examp. 1. To find the least common multiple of 7, 24, 40, 45, and 72.

$$
\begin{array}{r|rrrrr}
2) & 7 & 24 & 40 & 45 & 72 \\
2) & 7 & 12 & 20 & 45 & 36 \\
2) & 7 & 6 & 10 & 45 & 18 \\
3) & 7 & 3 & 5 & 45 & 9 \\
3) & 7 & 1 & 5 & 15 & 3 \\
5) & 7 & 1 & 5 & 5 & 1 \\
 & 7 & 1 & 1 & 1 & 1
\end{array}
$$

Then $2\times2\times2\times3\times3\times5\times7=2520$ is the multiple required; or the least number divisible by 7, 24, 40, 45, and 72.

Examp. 2. Required the least common multiple of 27, 66, 135, 275, and 675.

3)	27	66	135	275	675
3)	9	22	45	275	225
3)	3	22	15	275	75
5)	1	22	5	275	25
5)	1	22	1	55	5
11)	1	22	1	11	1
	1	2	1	1	1

Then $3 \times 3 \times 3 \times 5 \times 5 \times 11 \times 2 = 14850$ the multiple sought.

47. When the least denominator of two fractions exactly divides the greatest, multiply the terms of that fraction which hath the least denominator by the quotient.

Thus [illegible] and [illegible] are brought to a common denominator by multiplying the numerator and denominator of [illegible] by 2 (the quotient of 8 divided by 4).

And [illegible], [illegible], [illegible] are brought to a common denominator by multiplying the terms of [illegible] by 4, and those of [illegible] by 2, the three required fractions.

Or thus:

48. Having reduced the given fractions to their lowest terms, find the least common multiple of the denominators, which divide by the denominators, and multiply the numerators by the corresponding quotients; then the products placed over the said multiple give the fractions in their lowest terms.

Thus, let it be required to reduce the fractions [illegible], [illegible], and $\frac{10}{121}$, to equivalent fractions having the least common denominator.

The least common multiple of 14, 22, and 121 is 1694.

$$\left.\begin{array}{l} \frac{1694}{14} = 121 \\ \frac{1694}{22} = 77 \\ \frac{1694}{121} = \text{[illegible]} \end{array}\right\}$$ the three quotients or multipliers.

$$\left.\begin{array}{l} \text{Then } 3 \times 121 = 363 \\ \quad\;\; 5 \times 77 = 385 \\ \quad\; 10 \times 14 = 140 \end{array}\right\} \text{the three numerators.}$$

And $\frac{363}{1694}$, $\frac{385}{1694}$, $\frac{140}{1694}$, are the fractions required.

ADDITION OF VULGAR FRACTIONS.

49. Reduce compound fractions to simple ones; and all the fractions to a common denominator. Then add the numerators together and place the sum over the common denominator for the answer.

When the fractions are large, or numerous, it will be best to reduce them to the least common denominator.

Examp. 1. What is the sum of $\frac{1}{4}$, $\frac{2}{4}$, and $\frac{3}{4}$?

$1 + 2 + 3 = 6$. *Ans.* $\frac{6}{4}$ or $1\frac{1}{2}$.

2. Required the sum of $\frac{2}{11}$, $\frac{4}{11}$, and $\frac{5}{11}$?

$2 + 4 + 5 = 11$. *Ans.* $\frac{11}{11}$ or 1.

3. What is the sum of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$?

The fractions when brought to a common denominator will be $\frac{20}{60}$, $\frac{15}{60}$, and $\frac{12}{60}$.

$20 + 15 + 12 = 47$. *Ans.* $\frac{47}{60}$.

4. Required the sum of $\frac{6}{7}$, and $\frac{2}{3}$ *of* $\frac{3}{4}$?

$\frac{2}{3}$ *of* $\frac{3}{4} = \frac{6}{12} = \frac{1}{2}$;

$\frac{1}{2}$ and $\frac{6}{7}$ brought to a common denominator are $\frac{7}{14}$ and $\frac{12}{14}$: then $7 + 12 = 19$.

Ans. $\frac{19}{14} = 1\frac{5}{14}$.

50. When mixed numbers, or mixed numbers and fractions are to be added together, bring the fractions to a common denominator, then set down the integers as in common addition, and the fractions on the right hand:

Add the fractions together, and carry the integers (if any) from

the sum, to the numbers on the left, which add up as in common addition.

Ex. 5. What is the sum of $421\frac{7}{9}$, $67\frac{5}{9}$, and $\frac{8}{9}$?

$$\begin{array}{r} 67\frac{5}{9} \\ \frac{8}{9} \\ \hline 490\frac{2}{9} \end{array} \quad \textit{Answer.}$$

6. Required the sum of $1000\frac{2}{5}$, $74\frac{5}{9}$, and $6\frac{2}{45}$?

The fractions $\frac{2}{5}$, $\frac{5}{9}$, $\frac{2}{45}$ when brought to a common denominator are $\frac{18}{45}$, $\frac{25}{45}$, $\frac{2}{45}$:

$$\begin{array}{rr} & 1000\frac{18}{45} \\ & 74\frac{25}{45} \\ & 6\frac{2}{45} \\ \hline \text{Sum} & 1081 \end{array}$$

SUBTRACTION OF VULGAR FRACTIONS.

51. LET the fractions be prepared the same as for Addition: then the difference of the numerators set over the common denominator will give the difference of the proposed fractions.

Ex. 1. What is the difference of $\frac{1}{4}$ and $\frac{3}{4}$?

The difference of the numerators 1 and 3 is 2; therefore the required difference is $\frac{2}{4}$ or $\frac{1}{2}$.

2. Required the difference of $\frac{11}{19}$ and $\frac{2}{19}$?

$$\frac{11}{19} - \frac{2}{19} = \frac{11-2}{19} = \frac{9}{19} \quad \textit{Ans.}$$

3. Required the difference of $\frac{4}{5}$ and $\frac{5}{6}$?

$\frac{4}{5}$ and $\frac{5}{6}$ brought to a common denominator, are $\frac{24}{30}$ and $\frac{25}{30}$; therefore $\frac{25}{30} - \frac{24}{30} = \frac{1}{30}$. *Ans.*

4. What is the difference of $\frac{7}{9}$ and $\frac{2415}{3105}$?

$\frac{7}{9}$ and $\frac{2415}{3105}$ reduced to a common denominator are $\frac{21735}{27945}$ and $\frac{21735}{27945}$; *therefore the fractions are equal.*

5. From $\frac{4}{5}$ *of* $\frac{5}{6}$ take $\frac{2}{3}$ *of* $\frac{3}{4}$.

$\frac{4}{5}$ *of* $\frac{5}{6} = \frac{20}{30} = \frac{2}{3}$. $\frac{2}{3}$ *of* $\frac{3}{4} = \frac{6}{12} = \frac{1}{2}$.

$\frac{2}{3}$ and $\frac{1}{2}$ reduced to a common denominator are $\frac{4}{6}$ and $\frac{3}{6}$,

hence $\frac{4}{6} - \frac{3}{6} = \frac{1}{6}$ *Ans.*

52. When the difference of two mixt numbers, or a mixt number and a fraction is required, bring the fractions to a common denominator as before; then place the less number under the greater and take their difference for the answer. But if the lower fraction is greater than the upper one, subtract the numerator of the former from the sum of the terms of the latter, then set down the difference for the numerator of the remaining fraction, and carry 1 to be subtracted.

Ex. 6. From 74[illegible] 7. From 59101[illegible]
Take 16[illegible] Take [illegible]
Rem. 58[illegible] *Rem.* 5177[illegible]

8. Required the difference of 17[illegible] and [illegible]

The fractions [illegible] reduced to a common denominator [illegible]

[illegible] From 1012[illegible]
Take 56[illegible]
Rem. [illegible] In this example I take [illegible] from $\frac{11}{14}$ or 1. And in the preceding example, 7 is taken from 18 (the sum of the terms of the fraction [illegible]), which is the same thing as subtracting $\frac{7}{14}$ from $\frac{4}{14}$ added to $\frac{14}{14}$, [illegible], and evidently for the same reason that we borrow 10 in the subtraction of whole numbers when the figure to be subtracted [illegible] above it.

53. The reason why fractions must be brought to a common denominator for the purposes of addition and subtraction, will be evident, if we consider that in order to compare their several values, it is necessary to exhibit them in like parts of the integer.

Thus to compare $\frac{2}{3}$ with [illegible], if we suppose the integer 1 to be divided into

12 equal parts, $\frac{2}{3}$ will be $\frac{8}{12}$, and $\frac{3}{4}$ will be $\frac{9}{12}$; now the values being expressed in *12ths* (instead of *3ds* and *4ths*) it appears that $\frac{2}{3}$ is less than $\frac{3}{4}$ by $\frac{1}{12}$: also, that both together make $\frac{17}{12}$.

MULTIPLICATION OF VULGAR FRACTIONS.

54. REDUCE mixt numbers to improper fractions; and whole numbers to the form of fractions, by putting 1 for the denominators. Then multiply the numerators together for the numerator, and the denominctors together for the denominator of the product. This rule is the same as that for reducing a compound fraction to a simple one; for when the multiplier is a fraction, the product will be a part or parts *of* the multiplicand: thus $\frac{1}{2}$ *of* $\frac{1}{2}$ is $\frac{1}{4}$ or $\frac{1 \times 1}{2 \times 2}$; and $\frac{2}{3}$ *of* $\frac{3}{4}$ is $\frac{2}{4}$ or $\frac{2 \times 3}{3 \times 4}$; and therefore the fractions to be multiplied may be set down in the form of a compound fraction, aud the product found in the same manner as that is reduced to a simple one.

Examp. 1. What is the product of $\frac{3}{7}$ and $\frac{5}{8}$?

$$\frac{3 \times 5}{7 \times 8} = \frac{15}{56} \quad \textit{Ans.}$$

2. Required the product of $\frac{4}{9}$ and $\frac{18}{19}$?

$$\frac{4 \times 18}{9 \times 19} \quad \frac{4 \times 2 \times 9}{9 \times 19} \quad \frac{4 \times 2}{19} = \frac{8}{19} \quad \textit{Ans.}$$

3. What is the continued product of 4, $7\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$ of $\frac{6}{7}$?

First $4 = \frac{4}{1}$; and $7\frac{1}{2} = \frac{15}{2}$.

Then,

$$\frac{4 \times 15 \times 2 \times 5 \times 6}{1 \times 2 \times 3 \times 6 \times 7} \quad \frac{4 \times 15 \times 5}{3 \times 7} \quad \frac{4 \times 3 \times 5 \times 5}{3 \times 7} = \frac{4 \times 5 \times 5}{7} = \frac{100}{7} = 14\tfrac{2}{7} \quad \textit{Ans.}$$

4. What is $\frac{2}{3}$ of 29?

$\frac{2}{3} \times \frac{29}{1} = \frac{58}{3} = 19\frac{1}{3}$ the *answer*. Therefore to find the product of a fraction and a whole number, multiply by the numerator, and divide by the denominator.

55. When one factor is a whole, and another a mixt number; or if one is a small fraction, and another a large mixt number, multiply the parts of the latter separately, and add the products together.

Ex. 5. Required the product of $6742\frac{5}{121}$ by 8?

$$\begin{array}{r} 6742\frac{5}{121} \\ 8 \\ \hline 53936\frac{40}{121} \end{array} \quad Ans.$$

6. What is the product of $597\frac{4}{5}$ and 24?

$$\begin{array}{rcl} 597 \times 24 &=& 14328 \\ \frac{4}{5} \times 24 &=& 19\frac{1}{5} \\ \text{Sum} && 14347\frac{1}{5} \end{array} \quad Ans.$$

7. What is $\frac{3}{4}$ of $9614273\frac{5}{219}$?

$$\begin{array}{r} 9614273 \\ 3 \\ \hline 4\;)\,28842819 \\ \hline 7210704\frac{3}{4} \end{array} \qquad \frac{3}{4} \text{ of } \frac{5}{219} = [illegible]$$

$7210704\frac{3}{4} + [illegible] = 7210704\frac{127}{156}$ *Ans.*

56. And when both factors are mixt numbers, the product may be found by multiplying the parts separately, as in the next example.

Ex. 8. Required the pr[illegible] of [illegible] by [illegible]?

[illegible] × [illegible] = [illegible]
[illegible] × [illegible] = [illegible]
[illegible] × [illegible] = [illegible]
[illegible] × [illegible] = [illegible]
Sum [illegible] *Ans.*

DIVISION OF VULGAR FRACTIONS.

57. Prepare the fractions the same as for multiplication; then divide the terms of the dividend by the respective terms of the divisor, if they will exactly divide; but if not, then invert the divisor and proceed as in multiplication.

When the terms exactly divide, the truth of the rule is manifest from the principles of common division. And the reason for inverting the divisor in the other case will be evident if we consider that division is the reverse of multiplication: thus the product of $\frac{1}{2}$ and 4 is $\frac{1}{2} \times \frac{4}{1} = 2$ or the half of 4; but 4 divided by $\frac{1}{2}$ will give 8, because $\frac{1}{2}$ is contained 8 times in 4, the quotient being $\frac{2}{1} \times \frac{4}{1}$, where $\frac{2}{1}$ is the divisor $\frac{1}{2}$ inverted.

As a second example, let $\frac{5}{7}$ be divided by $\frac{2}{3}$; or suppose it is required to find how often $\frac{2}{3}$ is contained in $\frac{5}{7}$. Now if we divide 5 by $\frac{1}{3}$, the quotient will be $\frac{3}{1} \times \frac{5}{1}$ or 15, (because $\frac{1}{3}$ is contained 15 times in 5); but when the divisor is twice $\frac{1}{3}$, or $\frac{2}{3}$, the quotient will be only $\frac{1}{2}$ of 15, or $\frac{3 \times 5}{2}$ the quotient of 5 divided by $\frac{2}{3}$, consequently the 7*th.* of 5 (or $\frac{5}{7}$) will give but a 7*th.* of that quotient, or $\frac{3 \times 5}{2 \times 7}$; therefore the quotient $\frac{5}{7}$ divided by $\frac{2}{3}$ is truly expressed by $\frac{3 \times 5}{2 \times 7}$ equal to $1\frac{1}{14}$.

Ex. 3. Divide $\frac{14}{21}$ by $\frac{2}{7}$

$\frac{2}{7}$) $\frac{14}{21}$ ($\frac{7}{3}$ *quotient or answer.*

4. Required the 5*th* part of $\frac{10}{13}$:

$\frac{5}{1}$) $\frac{10}{13}$ ($\frac{2}{13}$ *Ans.*

5. Divide $\frac{9}{25}$ by $\frac{7}{5}$

$$\frac{5}{7} \times \frac{9}{25} = \frac{5 \times 9}{7 \times 5 \times 5} = \frac{9}{7 \times 5} = \frac{9}{35} \textit{ Ans.}$$

6. Divide $\frac{3}{4}$ *of* $\frac{4}{5}$ by $\frac{6}{5}$ *of* $\frac{4}{7}$

The divisor $\frac{6}{5}$ *of* $\frac{4}{7}$ when inverted is $\frac{5}{6} \times \frac{7}{4}$;

$$\frac{5 \times 7 \times 3 \times 4}{6 \times 4 \times 4 \times 5} = \frac{7 \times 3}{6 \times 4} = \frac{7 \times 3}{3 \times 2 \times 4} = \frac{7}{2 \times 4} = \frac{7}{8} \textit{ Ans.}$$

7. Let $3\frac{2}{5}$ be divided by $4\frac{5}{6}$

$3\frac{2}{5} = \frac{17}{5}$, and $4\frac{5}{6} = \frac{29}{6}$;

$\frac{6}{29} \times \frac{17}{5} = \frac{102}{145}$ *Ans.*

8. Divide 1 by $\frac{3}{7}$

$\frac{7}{3} \times \frac{1}{1} = \frac{7}{3}$ *quotient; this is called the reciprocal of the divisor* $\frac{3}{7}$.

9. Divide $\frac{5}{8}$ by 3

$\frac{1}{3} \times \frac{5}{7} = \frac{5}{21}$ *quotient.* Therefore to divide a fraction by a whole number, multiply the denominator by that number, except it will divide the numerator, as in *Ex.* 4.

58. If the divisor is a *whole,* and the dividend a large *mixt number,* divide the parts separately, and then add the quotients together.

Ex. 10. Required the *5th.* part of $4561412\frac{3}{7}$?

5)	4561412	
	$912282\frac{2}{5}$	the integral part divided by 5.
	$\frac{3}{35}$	the fraction $\frac{3}{7}$ divided by 5.
Sum	$912282\frac{17}{35}$	the *answer.*

59. When the divisor is a small fraction and the dividend a large mixt number, multiply the latter (without reducing it to an improper fraction) by the denominator of the divisor, and divide the product by the numerator.

Ex. 11. Divide $6421078\frac{3}{11}$ by $\frac{5}{6}$?

	$6421078\frac{3}{11}$	
	6	
5)	$38526469\frac{7}{11}$	product by the denominator 6.
	$7705293\frac{4}{5}$	the whole number divided by 5.
	$\frac{7}{55}$	the fraction $\frac{7}{11}$ divided by 5.
Sum	$7705293\frac{51}{55}$	*Ans.*

60. In like manner the quotient is found in the contracted method of division of whole numbers when the divisor is the product of two or more factors. (30. *Ex.* 11.)

Ex. 12. Let 8783 be divided by 56, or 7 times 8.

7)	8783	
8)	$1254\frac{5}{7}$	quotient by 7.
	$156\frac{6}{8}$	the whole number divided by 8.
	$\frac{5}{56}$	the fraction $\frac{5}{7}$ divided by 8.
Sum	$156\frac{47}{56}$	*Ans.*

OF DECIMALS.

61. DECIMALS are Fractions in the form of whole numbers, but whose values decrease from the place of units progressively to the right hand in the same decuple or tenfold proportion as the common scale of whole numbers increase to the left. They are usually separated from the integers by a comma or dot, the decimals being on the right hand.

Thus the mixt number $21\frac{2}{10}$ when the fraction is set down decimally will be 21·2; the 2 on the right of the 1, or dot, denotes 2 *tenths*, whereas the other 2 on the left are 2 *tens*. Another 2 on the left will be 2 *hundreds*, but on the right 2 *hundredths*, ($\frac{2}{100}$), and the whole or 221·22 is $221\frac{22}{100}$, because $\frac{2}{10}$ and $\frac{2}{100}$ together make $\frac{22}{100}$. A third figure on the left will be *thousands*, but on the right, the like number of *thousandth parts*. Thus 5008·005 is the same as $5008\frac{5}{1000}$; and 5000·005 the same as $5000\frac{5}{1000}$. Consequently a decimal fraction has always either 10, 100, 1000, &c. for its denominator; *viz.* the number of equal parts into which the *integer* or *whole* is supposed to be divided. For example, let a foot in length be the integer, and conceive it to be divided into 100 equal parts; then ·25 (or 25 with a dot on the left) will be the decimal part of a foot denoting 3 inches or $\frac{1}{4}$ ($\frac{25}{100}$ being $= \frac{1}{4}$). And $1\frac{1}{2}$ inches or $\frac{1}{8}$ of a foot will be ·125 of a foot, because 125 is $\frac{1}{8}$ of 1000; the foot in this case is supposed to be divided into 1000 equal parts.

Therefore to read, or set down a proposed decimal, it is only necessary to remember that the denominator is 1 with as many ciphers annexed as there are decimal places, or that the same number of figures to the right of the decimal point have always the same common denominator. Thus the denominator of the

fractions ·5000, ·0746, ·0005, is 10000. And hence it appears that the value of a decimal fraction is not altered by ciphers on the right hand; for ·5000 (or $\frac{5000}{10000}$) when reduced to its lowest terms is the same as ·5, each being equal to $\frac{1}{2}$.

ADDITION AND SUBTRACTION OF DECIMALS.

62. **Place** the numbers so that the decimal points may stand directly under each other; then add, and subtract, as in whole numbers, and set the decimal point in the sum or difference directly under the points above.

Ex. 1. Required the sum of ·7, ·014, and ·1246?

```
      ·7
      ·014
      ·1246
Sum   ·8386
```

By placing the decimal points under each other, tenths are brought under tenths, hundredths under hundredths, &c. whence the method of addition becomes the same as that for whole numbers.

The decimals in the foregoing example [illegible] written as vulgar fractions, are $\frac{7}{10}$, $\frac{14}{1000}$, and $\frac{1246}{10000}$, and when brought to a common denominator they will be $\frac{7000}{10000}$, $\frac{140}{10000}$, and $\frac{1246}{10000}$;

```
hence 7000
       140
      1246
      8386
```

8386 the sum of the numerators, and $\frac{8386}{10000}$, the sum of the fractions as before: but this is evidently nothing more than reducing the decimals to a common denominator by annexing ciphers on the right hand:

```
Thus  ·7000
      ·0140
      ·1246
Sum   ·8386
```

Ex. 2. What is the sum of ·0159, 54·77 and 9·299?

```
      ·0159
     54·77
      9·299
Sum  64·0849
```

Ex. 3 Required the sum of 9 *tenths*, 19 *hundredths*, 18 *thousandths*, 211 *hundred thousandths*, and 19 *millionth parts*?

$$\begin{array}{r} \cdot9 \\ \cdot19 \\ \cdot018 \\ \cdot00211 \\ \cdot000019 \\ \hline \text{Sum } 1\cdot110129 \end{array}$$

4. Required the difference of ·406 and ·11?

$$\begin{array}{l} \cdot406 \\ \cdot11 \\ \hline \text{[illegible]} \quad \textit{Ans.} \end{array}$$

5. [illegible] ·9901 and ·9078?

[illegible] *Ans.*

6. What is the difference of [illegible] and [illegible]?

[illegible] *Ans.*

7. Required the difference [illegible] and 24·98?

[illegible] *Ans.*

MULTIPLICATION OF DECIMALS.

63. Multiply as in whole numbers, and point off as many places for decimals in the product as there are decimals in both multiplier and multiplicand; but if there should not be so many, put ciphers on the left to supply the defect.

Ex. 1. Required the product of [illegible] and ·03?

·03

[illegible] *Ans.*

The decimals ·2 and ·03 when set down as vulgar fractions will be $\frac{2}{10}$ and $\frac{3}{100}$, and their product $\frac{2}{10} \times \frac{3}{100} = \frac{6}{1000}$ or 6 thousandth parts, as before. Hence the truth of the rule is evident.

Other Examples.

Multiply	·621
By	·26
	3726
	1242
Product	·16146

Multiply	·043
By	·003
Product	·000129

Multiply	621
By	·26
	3726
	1242
Product	161·46

Multiply	·0023
By	1700
	161
	23
Product	3·9100

Multiply	612
By	10000
Product	6120000.

Therefore multiplying by 10, 100, 1000, &c. is only removing the decimal point so many places to the right as there are ciphers in the multiplier. Thus 82·1 multiplied by 10 is 821; 1·4 multiplied by 1000 is 1400, &c.

64. There is a method of contracting the operation so as to retain only a proposed number of decimals in the product. Let ·5849 be multiplied by 7·26, and the product have only 3 decimal places.

·5849

4094

4·246 The required product is 4·246. But to omit setting down the figures on the right of the perpendicular bar, yet retain the product to the left, it is evident that the multiplication by the integer 7 must begin at 4 in the multiplicand or the 3d. place in the decimal from the left (3 being the number of decimals to be retained); the multiplication by 2 must begin at the 8; and that by 6 at the 5, remembering to carry from the figures omitted on the right hand, as in common multiplication. But when the figures of the multiplier are set down

in a contrary order, and the units place (7) is under (4) the 3*d* decimal from the left, the figures in the multiplier will stand directly under those in the multiplicand where the respective multiplications must begin.

```
 ·5849
 62·7
 ─────
 4094
  117
   35
 ─────
 4·246
```

Here 6 is carried to 7 times 4, because 7 times 9 (the figure omitted) is 63;—1 is carried to 2 times 8, because twice 4 (the figure on the right of 8) exceeds half 10.—And 5 is carried to 6 times 5, since 6 times 8 make almost 5 tens.

As a further illustration of this method of contraction, take the following examples.

Multiply 8467·73912 by 0·725184, reserving only 4 decimals in the product.

```
 8467·73912
  481527·0
 ──────────
 59274174
  1693548
   423387
     8468
     6774
      338
 ──────────
 6140·6689 Product.
```

Here (0) the units place stands under the 4*th* decimal from the left.

Multiply 3842·63 by 79·6543, retaining the integers only.

```
 3842·63
 3456·97
 ───────
 268984
  34584
   2306
    192
     15
      1
 ───────
 306082 Product.
```

Here units stand under units, no decimals being required in the product.

DIVISION OF DECIMALS.

65. DIVIDE as in whole numbers and point off as many decimals in the quotient as the number of decimals in the dividend exceed those in the divisor. But if the number of figures in

the quotient are not so many as the rule requires, prefix ciphers on the left to supply the defect.

If the number of decimals in the divisor exceed those in the dividend, annex ciphers to the latter before you begin the division.

When the divisor is 1 with ciphers on the right hand, remove the decimal point in the dividend as far to the left as there are ciphers.—But when the divisor is any other number with ciphers annexed, first divide by 10, 100, or 1000, &c. according to the number of ciphers; then divide the quotient by the remaining figure or figures. (60)

N. B. Should there be a remainder, after division, ciphers may be annexed to it, and the division continued as far as is necessary.

Examples.

Divide ·01728 by 14·4?

14·4) ·01728 (·0012 *quotient.*
144
288
288

Proof
14·4
·0012
Product ·01728 Hence it appears that the number of decimals in the divisor and quotient must be equal to those in the dividend; and therefore the truth of the rule is manifest.

Divide 17·28 by 14·4?

14·4) 17·28 (1·2 *quotient.*
144
288
288

Divide ·2123 by ·84?

·84) ·2123 (·25 &c. *quotient*
168
443
420
23

Divide 172·8 by ·144?

·144) 172·800 (1200 *quotient.*
144
288
288

Divide 192 by 5·423?

5·423) 192·0000 (35·4 &c. *quotient.*
162 69
29 310
27 115
2 1950
2 1692
258

Divide 548·3 by 10?

10) 548·3
54·83 *quotient.*

Divide 29·74 by 1000?

1000) 29·74
·02974 *quotient.*

Divide 6·48 by 200?

100) 6·48
2) ·0648 quotient by 100.
·0324 *Answer.*

Divide 64·9 by 7000?

1000) 64·9
7) ·0649 quotient by 1000.
·0092 &c. *Ans.*

Divide 591·27 by 470?

10) 591·27
47) 59·127 quotient by 10.
(1·2581 &c. *Ans.*
47
121
94
272
235
370
188
190
188
2

Divide 290·6 by 24000?

1000) 290·6
4) ·2906
6) ·07265
·0121083 &c. *Ans.*

66. When a certain number of decimals only are wanted in the quotient, the division may be contracted in the following manner:

Take the divisor one figure more than the number of figures required to be in the quotient.

Make each remainder a new dividend, and for every such dividend leave out a figure on the right hand of the divisor, remembering to carry for the increase of the figures omitted, as in the contraction of multiplication. (Art.

Let 2478 be divided by 5·46071381 so as to have decimals in the

The number of figures in the quotient will be 6, viz. 2 integers and 4 decimals, therefore we must take 7 figures for the divisor;

```
2·846712 | 81 ) 94·78000 ( 33·2945 quotient.
                8540138
       284671 )  937862
                 854014
        28467 )   83848
                  56934
         2846 )   26914
                  25620
          284 )    1294
                   1138
           21 )     156
                    142
                     14
```

The two right hand figures (81) of the given divisor are cut off, and 2 are carried for the product of 8 by 3. And instead of bringing down each divisor (as above) the figures may successively be pointed off. It is also evident when the number of figures in the divisor is less than the number required in the quotient, that ciphers must be added to the former.

To reduce a Vulgar Fraction to an equivalent Decimal.

67. Add ciphers to the numerator and divide by the denominator, then point off as many decimal places in the quotient for the answer as there were ciphers added. This is continuing the division of whole numbers when there is a *remainder*, by which means we get a *decimal* in the quotient instead of a vulgar fraction.

For example, if 97 be divided by 32, the quotient is $\frac{97}{32}$ or $3\frac{1}{32}$, but if ciphers are added we shall have 3·03125 for the quotient.

Thus,

```
32 ) 97·00000 ( 3·03125 quotient.
     96
      100
       96
        40
        32
         80
         64
         160
         160
```

The ciphers annexed only point out the number of decimal places, and therefore 97·00000 is the same as 97, consequently 97·00000 and 97 when divided by 32 must give equal quotients, and therefore the decimal ·03125 is equivalent to $\frac{1}{32}$. Which also will be evident by taking the decimal for a vulgar fraction, for $\frac{3125}{100000}$ reduced to its lowest terms is $\frac{1}{32}$.

Other Examples.

Reduce $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ to equivalent decimals?

4) 1·00 — ·25
2) 1·0 — ·5
4) 3·00 — ·75 *Ans.* ·25, ·5, and ·75.

Reduce $\frac{3}{625}$ to a decimal?

625) 3·0000 (·0048 *Ans.*
2500
5000
5000

Reduce $\frac{2}{12500}$ to a decimal?

12500) 2·00000 (·00016. *Ans.*
1 2500
75000
75000

Reduce $\frac{246}{10000}$ to a decimal?

10000) 246
·0246 *Ans.*

Reduce $\frac{10}{12}$ to a decimal?

12) 10·000 (·833 &c. *Ans.*
9 6
40
36
40
36
4

Divide 98 by 24000; or which is the same thing, reduce $\frac{98}{24000}$ to a decimal?

1000) 98
4) ·098
6) ·0245
·0040833 &c. *Ans.*

Reduce $\frac{1}{7}$ to a decimal?

7) 1·000 &c.
·1428571428 &c. *Ans.*

Reduce $\frac{1}{9}$ to a decimal?

9) 1·0000 &c.
·1111111 &c. *Ans.*

The decimals in the last examples are called circulating, recurring, or repeating decimals, because the same figure or figures are regularly repeated.

68. When an improper fraction is to be reduced, the answer will be a mixt number. Thus $\frac{9}{4}$ = 2·25; and $\frac{76}{6}$ = 12·666 &c.

69. If the right hand figure of the denominator of the fraction to be reduced is either 1, 3, 7, or 9, it is manifest there will always be a remainder after division, and consequently the decimal can never be brought exactly equal to the vulgar fraction: in such cases continue the division to as many places of figures as may be thought necessary.

OF DUODECIMALS.

70. DUODECIMALS are vulgar fractions whose denominators are 12, or 144, or 1728, &c. The division and subdivision of the integer being in 12*ths* instead of 10*ths* as in the decimal scale. The method of operating with these fractions (sometimes called *cross multiplication*) seems to have been introduced for the purpose of computing the contents of artificer's work when the dimensions are in *feet* and *inches*, or in *feet*, *inches*, and *lines* or 12*ths* of an *inch*.

71. *Rule.* Set down the dimensions to be multiplied together so that feet may stand under feet, inches under inches, &c.

Begin with the feet in the multiplier and the lowest dimension in the multiplicand. Carry 1 to the next for every 12 in each product, setting down the remainder directly under the corresponding terms of the multiplicand. Next, multiply by the inches, and carry 1 for every 12 as before, remembering to set down the result one place to the right of the product by the feet. And in the same manner proceed with a third dimension, &c. Then add the several products together, carrying 1 for every 12 as before, and the sum will be the answer.

	F.	In.	lines.
Exam. 1. Multiply	7	9	0
By	5	6	0
	38	9	0
	3	10	6
Product	42	7	6

Or $42 + \frac{7}{12} + \frac{6}{144} = 42$ [illegible] *Feet.*

The reason for setting down the products in the manner directed by the rule will be evident if we retain the denominators, but multiply by the parts separately as before (see examp. 8. in the multiplication of vulgar fractions):

Thus,

$$\begin{array}{rl}
 & \textit{Feet.} \\
\text{Multiply} & 7\tfrac{9}{12} \\
\text{By} & 5\tfrac{6}{12} \\
\hline
 & 38\tfrac{9}{12} \ldots\ldots\ldots \text{Product by } 5. \\
 & 3\tfrac{10}{12}\ \tfrac{6}{144} \ldots\ldots \text{Product by } \tfrac{6}{12}. \\
\hline
\textit{Product} & 42 + \tfrac{7}{12} + \tfrac{6}{144}
\end{array}$$

First, $\frac{9}{12} \times 5 = \frac{45}{12} = 4\frac{9}{12}$ or 3 to be carried and $\frac{9}{12}$ to set down, and $7 \times 5 = 35$ which with 3 make 38.

Secondly, $\frac{9}{12} \times \frac{6}{12} = \frac{54}{144}$ or $\frac{4}{12}$ to be carried and $\frac{6}{144}$ to set down (because 12 of any denomination make 1 of the next superior or that on the left):

And $7 \times \frac{6}{12} = \frac{42}{12}$ which with $\frac{4}{12}$ (that were carried) make $\frac{46}{12}$, or $3\frac{10}{12}$ to set down.

The answer however, in this example, is sooner obtained by the usual method of vulgar fractions, thus.

$$\begin{array}{lll}
\textit{F.} & \textit{in.} & \textit{F.} \\
7 & 9 & = \tfrac{31}{4} \\
5 & 6 & = \tfrac{11}{2}
\end{array}$$

and $\frac{31}{4} \times \frac{11}{2} = \frac{341}{8} = 42\frac{5}{8}$ Ans. as before.

Ex. 2. Required the number of square feet in a rectangular board whose length is 17 *F.* $5\frac{1}{2}$ *in.* and breadth 1 *F.* $8\frac{3}{4}$ *in.*

Ans. The two dimensions multiplied together give the answer.

$$\begin{array}{rrrrrl}
\textit{F.} & \textit{in.} & \textit{lines.} & & & \\
17 & 5 & 6 & & & \\
1 & 8 & 9 & & & \\
\hline
17 & 5 & 6 & & & \\
11 & 7 & 8 & 0 & & \\
1 & 1 & 1 & 1 & 6 & \\
\hline
\textit{Product } 30 & 2 & 3 & 1 & 6 & \text{or 30 and [illegible]}
\end{array}$$

Ex 3 Required the number of cubic feet in a rectangular ditch whose breadth is 8½, depth 4⅔, and length 121 feet?

Here the three dimensions must be continually multiplied together

F.	*In.*
4	8
8	6
37	4
2	4
39	8
121	0
4799	8, or $4799\frac{8}{12}$, *cubic feet*

Or thus,

121	
4⅔	
484	product by 4,
40⅓	product by ⅓
40⅓	ditto
564⅔	= 121 × 4⅔
8½	
4517⅓	product by 8
282⅓	product by ½
4799⅔	the answer, as before

72. # TABLES

OF MONEY, WEIGHTS AND MEASURES.

OF MONEY.

		Marked.
Farthings		qrs.
4 Farthings	= 1 Penny	d.
12 Pence	= 1 Shilling	s.
20 Shillings	= 1 Pound	
240 Pence, or 960 Farthings	= 1 £.	

PENCE TABLE.

d.		s.	d.
20	is	1	8
30		2	6
40		3	4
50		4	2
60		5	0
70		5	10
80		6	8
90		7	6
100		8	4
110		9	2
120		10	0

TROY WEIGHT.

		Marked.
Grains		gr.
24 Grains	= 1 Pennyweight	dwt.
20 Pennyweights	= 1 Ounce	oz.
12 Ounces	= 1 Pound	lb.

By this weight are weighed Gold, Silver, Jewels, and some Liquids. ... they sometimes express the weight of a diamond in carats of 4 grains (four grains weight) each. A carat, however, signifies the $\frac{1}{24}$ of any mass of gold, or of gold with alloy, and is generally used to denote its degree of fineness.

APOTHECARIES WEIGHT.

		Marked.
Grains		gr.
20 Grains	make 1 Scruple	sc. or ℈
3 Scruples	1 Dram	dr. or ʒ
8 Drams	1 Ounce	oz. or ℥
12 Ounces	1 Pound	lb. or ℔

The Pound is the same as the Pound Troy, only differently divided.

Apothecaries use this weight in compounding their medicines, but buy and sell their Drugs by Avoirdupois Weight.

AVOIRDUPOIS WEIGHT.

				Marked	
	Drams			dr.	
16	Drams	make	1 Ounce	oz.	gr.
16	Ounces		1 Pound	lb.	= 7000 Troy
28	Pounds		1 Quarter	qr	
4	Quarters, or 112lb.		1 Hundred Weight	cwt.	
20	Hundred		1 Ton	ton.	

And 8lb. is a Stone in the London Markets.
14lb. a Stone, Horseman's Weight.
28lb. a Tod

By this Weight are weighed all Groceries, Chandlers' Wares, &c. and all Metals except Gold and Silver.

LONG MEASURE.

3	Barley Corns	make	1 Inch
12	Inches		1 Foot.
3	Feet		1 Yard.
6	Feet		1 Fathom.
$5\frac{1}{2}$	Yards, or $16\frac{1}{2}$ Feet		1 Rod, Pole
40	Rods		1 Furlong
8	Furlongs, or 1760 Yards		1 Mile.
3	Miles		1 League.
$69\frac{1}{2}$	Miles (nearly)		1 Degree.
360	Degrees the Earth's circumference.		

Also,

4	Inches	make	1 Hand, or hands breadth
5	Feet		1 Geometrical Pace.
4	Poles, or 66 Feet, } in.		1 Chain.
100	Links, each $7\frac{92}{100}$ }		

CLOTH MEASURE.

$2\frac{1}{4}$	Inches	*make*	1 Nail.
4	Nails		1 Quarter of a Yard.
4	Quarters		1 Yard.
3	Quarters		1 Ell Flemish.
5	Quarters		1 Ell English.
6	Quarters		1 Ell French.

SQUARE MEASURE.

144	Square Inches	*make*	1 Foot Square.
9	Square Feet		1 Yard.
$30\frac{1}{4}$	Square Yards		1 Pole.
40	Square Poles		1 Rood.
4	Roods, or 160 Square Poles		1 Acre.
4840	Square Yards		1 Acre.
10	Square Chains		1 Acre.
[illegible]	Square [illegible]		1 Acre.

[illegible] which have length and breadth [illegible]

CUBIC OR SOLID MEASURE.

1728	Cubic Inches	*make*	1 [illegible]
27	Cubic Feet		1 Yard.

By this Measure, Stone, Timber [illegible] of three dimensions (length, breadth, and depth) are measured.

DRY OR CORN MEASURE.

2	Pints	*make*	1 Quart.
2	Quarts		1 Pottle.
2	Pottles, or 4 Quarts		1 Gallon

2 Gallons	....	1 Peck.
4 Pecks	....	1 Bushel.
8 Bushels	. ...	1 Quarter.
5 Quarters, or 40 Bushels	......	1 Load, or Weigh.
2 Weighs	....	1 Last.

The Corn or Winchester Bushel is 8 inches deep, and 18½ inches in diameter, and contains 2150⅖ cubic inches; therefore the gallon contains 268⅘. But the Coal Bushel must be 19½ inches in diameter, and 36 Bushels make a Chaldron.

ALE AND BEER MEASURE.

2 Pints	.... *make*	1 Quart.
4 Quarts	... .	1 Gallon.
36 Gallons	. ...	1 Barrel.
1½ Barrels	... .	1 Hogshead.
2 Barrels	....	1 Puncheon.
2 Hogsheads	....	1 Butt.
2 Butts	....	1 Tun.

The Ale Gallon contains 282 Cubic inches.

WINE MEASURE.

2 Pints	.... *make*	1 Quart.
4 Quarts	....	1 Gallon.
42 Gallons	....	1 Tierce.
63 Gallons	... .	1 Hogshead.
2 Tierces	. ...	1 Puncheon.
2 Hogsheads	. ...	1 Pipe, or Butt.
2 Pipes	....	1 Tun.

The Gallon contains 231 cubic inches.

By this are measured, all Wines, Spirits, Cyder, Honey, Oil, &c.

TIME.

60 Seconds	.... *make*	1 Minute.
60 Minutes	....	1 Hour.

24 Hours	....	1 Natural Day.
365 Days, 6 Hours	....	1 Julian Year.
365 *Days*, 5 *h*. 48 *min*. 45 *sec*.	..	1 Solar Year.

74. FOREIGN MEASURES OF LENGTH.

The Rhynland Foot	= 1·033	English Feet.
Rhynland Rood, 12 Rhynland Feet	= 12·396	

	Yards.
The French Toise, 6 Paris Feet	2·1315
Common French League, 2000 Toises	4263
Common French League, 25 to a degree ..	4869
Brabant League, 2800 Toises (nearly)	5968
Italian Mile, 60 to a Degree	2029
German Mile, 15 to a Degree	8116

The scales to the French and the German Military Maps and Plans are commonly in Leagues, Miles, Toises, or Rhynland Roods. But the "great" and "common" German Miles seem to be of no determinate length; according to the Table in the Field Engineer, they vary from 10020 to 28530 [illegible]. And we sometimes find a scale denominated, "a [illegible] hours walk on the road."

74. From the measurements lately carried on through France and part of Spain, the French Mathematicians conclude (according to a particular hypothesis) that ¼ of the whole terrestrial meridian is 5130740 *Toises* in length; and the *ten millionth part*, or ·5130740 of a *Toise* is the "*Metre*," or standard for the measures of length now adopted in France. This Metre is equal to [illegible]0832 English Feet.

OF REDUCTION.

75. The operation of changing numbers from one name or denomination to another without altering their value, is called *Reduction*.

76. When a greater denomination is to be reduced to a less (as pounds to shillings, or feet to inches) the process is by Multiplication. But less denominations are brought to greater by Division.

Ex. 1. Reduce £84 to shillings, pence, and farthings?

By the first of the foregoing tables it is evident that
Pounds multiplied by 20 give shillings.
Shillings multiplied by 12 give pence.
Pence multiplied by 4 give farthings.

Consequently,
Farthings divided by 4 give pence.
Pence divided by 12 give shillings.
Shillings divided by 20 give pounds.

```
  84
  20
----
1680 the shillings.
  12
----
20160 the pence.
   4
----
80640 the farthings.
```

Ex. 2. Reduce 80640 farthings to pounds?

```
 4 ) 80640
12 ) 20160  the pence.
20 )  1680  the shillings.
        84  the pounds.
```

Otherwise, as 960 farthings are equal in value to £1, if 80640 be divided by 960 the quotient will be the number of pounds required.

```
                 £
960 ) 80640 ( 84 as before.
      7680
      ----
       3840
       3840
```

Ex. 3. Reduce 26779 farthings to pounds?

```
 4 ) 26779
12 )  6694 — 3
20 )   557 — 10            £   s.  d.
        27 — 17        =   27. 17. 10¾
```

1. Reduce £ 27 17s. 10¾d. to farthings?

```
   27
   20
  540
   17   the 17s. add.
  557
   12
 6684
   10   the 10d. add.
 6694
    4
26776
    3   the 3 farthings add.
26779   farthings, the Answer.
```

2. Reduce 931 guineas to pounds?

```
     931
      21
     931
    1862
20 ) 19551   shillings.
      977—11          Ans. £977 11s.
```

3. Reduce £$\frac{4}{17}$ to farthings?

and $\frac{4}{17} \times 960 = \frac{3840}{17} = 225\frac{15}{17}$ *Ans.*

4. What is $\frac{2}{7}$ of a £?

```
      2
     20
7 )  40 ( 5 s.
     35
      5
     12   d.
7 )  60 ( 8
     56
      4
      4   qrs.                  s. d. qrs.
7 )  16 ( 2 2/7       Ans.      5  8  2 2/7
     14
      2
```

5. Reduce £$\frac{1}{320}$ to pence, or rather to the fraction of a penny?

$\frac{1}{320} \times 240 = \frac{240}{320} = \frac{24}{32} = \frac{3}{4}^{d}$ *Ans.* Therefore $\frac{1}{320}$ of a £ is equal to $\frac{3}{4}$ of a penny, or 3 farthings.

9. Reduce $\overset{d.}{5\frac{3}{4}}$ to the fraction of a sh

$\overset{d.}{5\frac{3}{4}}$ = 23 farthings, which divided by 48 (the farthings in a shilling) gives $\frac{23}{48}$ the *Answer*.

10. Reduce $\frac{2}{11}$ of a guinea to the denomination or fraction of a crown?

$\frac{2}{11} \times 21 = \overset{sh.}{\frac{42}{11}}$ which divided by 5 gives $\frac{42}{55}$ the *Answer*.

11. Reduce $\overset{£}{\cdot 93}$ to farthings.

$\cdot 93 \times 960 = 892\cdot 8$ farthings, the *Answer*.

12. What is ·885 of a £?—Or to find the value of the decimal of a pound.

$$\begin{array}{rl} & \cdot 885 \\ & 20 \\ \hline \text{Shillings} & 17\cdot 700 \\ & 12 \\ \hline \text{Pence} & 8\cdot 400 \\ & 4 \\ \hline \text{Farthings} & 1\cdot 600 \end{array}$$

Ans. 17

13. Bring 9·84 pence to the decimal of a £.

$$240\ )\ 9\cdot 840\ (\ \cdot 041\ \textit{Ans.}$$

960
240
240

77. In like manner other denominations are reduced by means of the numbers in the foregoing tables, remembering to multiply, or divide, as the case may require.

Ex. 14. How many guineas weigh a *lb.* Troy, each being

$12 \times 20 \times 24 = 5760 = 1\,lb.$

$\overset{dw.}{5}\ \overset{gr.}{9\frac{1}{2}} = \overset{gr.}{129\frac{1}{2}} = \overset{gr.}{\frac{259}{2}}$

5760 divided by $\frac{259}{2}$, is $\frac{2}{259} \times 5760 = \frac{11520}{259} = 44\frac{124}{259}$ *Ans.*

15. If 10000 men have each 40 rounds of cartridge with ball, what is the whole weight of lead, the balls being an ounce each?

$10000 \times 40 = 400000$,

lb.

$\frac{400000}{16} = 25000$.

C. lb. ton. C. lb.

$\frac{25000}{112} = 223\ 24 = 11\ 3\ 24$ *Ans.*

(Wh[illegible] an hundred weight?

[illegible]

lb. oz. dr.

Ans. 106 6 6·4

[illegible] Reduce [illegible] decimal of a yard?

3) 2·21 [illegible] &c. *Ans.*

[illegible] Reduce ⅜ of a [illegible] to yards, &c.?

3
1760
) 5280 (754
[illegible]
36
7) 72 ($10\frac{2}{7}$ *Ans.* *yds. in.* 754 $10\frac{2}{7}$

19. What is ·625 of a yard?

·625
3
1·875
12
10·500 *Ans.* *F. in.* 1 10½

20. Reduce 59·74 square inches to the decimal of a square foot?

$\frac{59 \cdot 74}{144} =$ ·4148 &c. *Ans.*

21. Reduce $\frac{5}{6}$ of a cubic yard to cubic feet?

$$\frac{5}{6} \times 27 = \frac{135}{6} = 22\tfrac{1}{2}.\ \textit{Ans.}$$

22. Reduce 64·984 cubic inches to the decimal of a cubic foot?

$$\frac{64\cdot984}{1728} = \cdot0376\ \&c.\ \textit{Ans.}$$

23. Reduce 500 Rhynland roods to English miles?

$$3)\underline{12\cdot396}$$
$$4\cdot132 \text{ yards} = 1 \text{ rood.}$$

$$4\cdot132 \times 500 = 2066 \text{ yards} = \overset{m.}{1}\ \overset{yds.}{306}\ \textit{Ans.}$$

24. Reduce an English mile to toises?

$$\frac{1760}{2\cdot1315} = 825\cdot709\ \&c.\ \textit{Ans.}$$

25. Reduce 5 French leagues (25 to a degree) to English miles?

$$\frac{4889 \times 5}{1760} = \overset{m.}{13}\ \overset{yds.}{14[illegible]5}\ \textit{Ans.}$$

COMPOUND ADDITION.

78. COMPOUND Addition is the collecting several numbers of different denominations into one sum.

79. *Rule.* Reduce fractional quantities of different denominations to like denominations. And fractions having different denominators to a common denominator. Then set down the numbers so that those of the same denomination may [illegible] directly under each other, as pounds under pounds, shillings under shillings, feet under feet, &c.

Add up the figures in the lowest denomination, and find [illegible] the rule of Reduction how many units of the next higher denomination are contained in the sum. Set down the [illegible] and carry the units to the next denomination, [illegible]

in the same manner as before; and so on till the whole is finished.

Examples.

1. Required the sum of £15 18s. 2¾d., £5 10s. 11¼d., £74 17s. 8½d., and £29 19s. 5¾d.

	£	s.	d.
	15	18	2¾
	5	10	11¼
	74	17	8½
	29	19	5¾
Sum	126	6	4¼

The number of farthings are 9, which make 2 pence to carry to the pence, and 1 farthing to set down. The pence in the next column are 26, and 2 carried make 28, or 4 pence over 2 shillings. The sum of the shillings 64, with 2 carried make 66, or 6 shillings to set down and 3 to carry to the pounds.

2. What is the sum of £4 16s. 9¼d., £5 0s. 0½d., and £10 3s. 2¼d.?

	£	s.	d.
	4	16	9¼
	5	0	0½
	10	3	2¼
Sum	20	0	0

3. Required the sum of £11 [illegible] and £0 17s. 9¾d.?

£ ⅕ = [illegible] s. 2 d. [illegible] qrs.

	£	s.	d.	qrs.
	11	[illegible]	[illegible]	2
	0	17	9	3
Sum	[illegible]	19	11	1

4. What is the sum of [illegible] and 19[illegible]?

[illegible]

[illegible] = 24[illegible]

10[illegible] = 19[illegible]

22 [illegible]

[illegible]

5. Add £2·29 and 17·211s. together?

```
 ·29
 ·20
 5·80
```

	£		s.
£ 2·29	=	2	5·8
			17·211
Sum		3	3·011

6. What is the sum of 77 guineas, 13 half guineas, three 15*l.* notes, 11 half-crowns, and 29 dollars at 4*s.* 1½*d.* each?

77 guineas....................	=	80	17	0
13 half-guineas..................	=	6	16	6
Three 15 *l.* notes..................	=	45	0	0
11 half-crowns..................	=	1	7	6
29 dollars, at 4s. 1½d........	=	[illegible]	0	[illegible]
		[illegible]	1	[illegible] *Ans.*

7. Add *lb. oz. dwts. gr.* 2 10 18 19, *oz. dwts. gr.* 11 15 16, and *lb. oz. dwts.* 3 4 14 together?

	lb.	*oz.*	*dwts.*	*gr.*
	2	10	18	19
		11	15	16
	3	4	14	0
Sum	7	3	8	11

8. Add ℔ ℥ ʒ ℈ *gr.* 6 10 7 2 19, ℔ ℥ ʒ ℈ *gr.* 1 6 6 2 18, and ʒ ℈ *gr.* 7 1 10 together.

℔	℥	ʒ	℈	*gr.*
6	10	7	2	19
1	6	6	2	18
		7	1	10
8	[illegible]	6	1	7 *Ans.*

9. Let *cwt. qr. lb. oz.* 2 1 27 15, *cwt. qr. lb. oz.* 5 3 26 14, and *qr. lb. oz. dr.* 3 10 12 15 be added together.

cwt.	*qr.*	*lb.*	*oz.*	*dr.*
2	1	27	15	0
5	3	26	14	0
	3	10	12	15
9	1	9	9	15 *Ans.*

10. Let 4·56 cwt. and 104·44 lb. be added together?

cwt. lb.
·56 × 112 = 62·72

cwt.	lb.	
4	62·72	
0	104·44	
5	55·16	*Ans.*

11. Add ·262 yd., 2·4 feet, and $\frac{7}{9}$ of an inch together?

		inch.
yd. ·262 × 36	=	9·432
feet. 2·4 × 12	=	28·8
in. $\frac{7}{9}$	=	0·7777, &c.
	Inches	39·0097 *Ans.*

12. The contents of three fields A, B, C, were as below; required the whole number of acres?

		ac.	roods	pol.
viz.	A	14	3	37
	B	16	3	30
	C	10	2	24
		42	2	11 *Ans.*

13. A field having been measured with the chain in 4 divisions, the contents were found as below. Required the whole number of acres?

	chains	[illegible]
viz.	43	[illegible]
	23	6[illegible]3
	14	[illegible]
	91	546[illegible]
Acres	17 3	·9070 *Ans.*

14. The several contents of a piece of work are 24*f.* 124*in.* 14*f.* 100*in.* 99*f.* 29*in.* and 16*f.* 99*in.*; what is the whole content?

F.	in.	
24	124	
14	100	
99	29	
16	99	
155	64	*Ans.*

15. The cubic contents of three pieces of timber are 29*f.* 1629*in.* 24*f.* 1561*in.* and 19*f.* 1104*in.* how many feet in the whole?

F.	*in.*	
29	1629	
24	1561	
19	1104	
74	838	*Ans.*

16. Add $3\frac{3}{7}$ cubic yards, and $21\frac{1}{3}$ cubic feet together?

$$\overset{yd.}{\tfrac{3}{7}} \times 27 = \overset{f.}{\tfrac{81}{7}} = 11\tfrac{4}{7} = \overset{f.}{11\tfrac{12}{21}}; \text{ and } \overset{f}{21\tfrac{1}{3}} = \overset{f.}{21\tfrac{7}{21}}$$

	yd.	*f.*
	3	$11\frac{12}{21}$
		$21\frac{7}{21}$
Sum	4	$5\frac{19}{21}$

COMPOUND SUBTRACTION.

60. *Rule.* Prepare the numbers and set them down as in Addition, only let the less stand under the greater.

Begin at the right hand, and take each number in the lower line from that above it and set the remainder directly under: but if any number in the lower line be greater than that above it, instead of adding 10 to the upper one, as in simple subtraction, increase it by as many as make one of the next higher denomination, then subtract the lower number from the sum, and set down the remainder. Carry 1 for that borrowed to the next number in the lower line, and proceed as before till the whole is finished.

Examples.

	£	*s.*	*d.*
1. From	54	16	3
Take	19	4	2
Rem.	35	12	1
Proof	54	16	3

	£	*s.*	*d.*
2. From	19	0	$9\frac{3}{4}$
Take	6	0	$9\frac{1}{2}$
Rem.	13	0	$0\frac{1}{4}$
Proof	19	0	$9\frac{3}{4}$

	£	*s.*	*d.*
3. From	5	2	$1\frac{1}{4}$
Take	1	3	$2\frac{1}{2}$
Rem.	3	18	$10\frac{3}{4}$

Here 3 farthings being greater than 1 farthing, I borrow 1 penny or 4 farthings, which added to $\frac{1}{4}$ in the upper line make 5 farthings, then 3 from 5 leave 2 farthings or $\frac{1}{2}$ a penny to set down. Next, 1 that was borrowed and 2 make 3, which taken from 13 (because I borrow 1*s.* or 12 pence, and add it to the 1 in the upper line) and 10 remains. Carrying 1 that was borrowed to the 3 shillings and the sum is 4, which subtracted from 22 (because I borrow 20) leaves 18. Lastly, 1 carried to 1, and the sum taken from 5 gives 3 the last remainder.

	£	s.	d.
4. From	27	0	0
Take	25	19	$11\frac{1}{2}$
Rem.	1	0	$0\frac{1}{2}$

	£	s.	d.
5. From	1	0	$0\frac{1}{4}$
Take	0	0	$0\frac{1}{2}$
Rem.	0	19	$11\frac{3}{4}$

6. What is the difference of £$\frac{1}{7}$ and $\frac{1}{7}$*s.*?

$$\frac{1}{7} \times 20 = \frac{20}{7} = 2\frac{6}{7}s. = £\frac{1}{7}.$$

$$\begin{array}{r} 2\frac{6}{7} \\ \frac{1}{7} \\ \hline \textit{Shill.}\quad 2\frac{5}{7}\ \textit{Ans.} \end{array}$$

7. What is the difference of $\frac{2}{9}$ of a guinea and $\frac{7}{30}$ of a pound?

$\frac{2}{9}$ of a guinea $= \frac{2}{9} \times \overset{s.}{21} = \frac{42}{9} = 4\frac{2}{3}$ *shillings.*

$\overset{£}{\frac{7}{30}} = \frac{7}{30} \times \overset{s.}{20} = \frac{140}{30} = 4\frac{2}{3}$ *shillings.* Therefore the difference is nothing.

8. Required the difference between £0·252 and 5·218*s.*?

$$\begin{array}{rr} & £ \\ & 0·252 \\ & 20 \\ \hline \textit{Shil.} & 5·040 \end{array} \qquad \begin{array}{r} s. \\ 5·218 \\ 5·04 \\ \hline 0·178 \end{array}\ \textit{Ans.}$$

	yd.	f.	in.
9. From	17	1	$4\frac{1}{2}$
Take	15	2	11
Diff.	1	1	$5\frac{1}{2}$

	yd.	f.	in.
10. From	6	0	4·75
Take	2	1	5·84
Diff.	3	1	10·91

11. From $11\frac{1}{7}$ square feet take $100\frac{1}{7}$ square inches.

$\frac{1}{7} \times 144 = \frac{144}{7} = 20\frac{4}{7}$ square inches.

	F.	*in.*	
	11	$20\frac{4}{7}$	
	0	$100\frac{1}{7}$	
Diff.	10	$64\frac{3}{7}$	*Ans.*

Or thus,

$$\overset{in.}{100\tfrac{1}{7}} = \overset{f.}{\frac{701}{7 \times 144}} = \overset{f.}{\frac{701}{1008}}.$$

$$\overset{feet}{11\tfrac{1}{7}} = \overset{f.}{11\tfrac{144}{1008}}$$

$0\frac{701}{1008}$ sub.

$10\frac{451}{1008}$ the *answer* in square feet.

12. What is the difference between ·58 of a solid yard and 16·66 solid feet?

$$\cdot 58 \times 27 = \overset{feet}{15\cdot 66}$$

16·66

Diff. 1

COMPOUND MULTIPLICATION AND DIVISION.

81. Compound Multiplication and Division are compendious methods of Compound Addition and Subtraction.

I. *When the multiplier is a whole number.*

82. *Rule.* Multiply the number in the lowest denomination, and find, by the rule of Reduction, how many integers of the next superior denomination are contained in the product, and set down the remainder if any. Carry the integers thus found to the product of the next higher denomination, with which proceed as before till the whole is multiplied.

II. When the divisor is a whole number.

83. *Rule.* Divide the highest denomination by the divisor and set down the quotient; and if there be any remainder, find how many integers of the next denomination it is equal to, and add them to the number (if any) which stands in that denomination. Divide the number thus found by the divisor, and set down the quotient under its proper denomination. Reduce the remainder to the next lower denomination; and so on, till the whole is finished.

Examples in Multiplication by whole numbers.

1. What cost 7 quarters of oats at £1 9s. 10d. per quarter?

£	s.	d.	
1	9	10	
		7	
10	8	10	*Ans.*

2. At £2 19s. 10½d. per barrel, what is the cost of 10 barrels of gunpowder?

£	s.	d.	
2	19	10½	
		10	
29	18	9	*Ans.*

3. What is the whole length of 9 planks, each being 13 f. 5 in. 7?

f.	in.		
13	5	7	
		9	
121	2	3	*Ans.*

84. When the multiplier is the product of two or more single figures, the answer may be found by multiplying successively by those figures instead of the whole at once. (19)

4. What cost 25 chaldron of coals at £2 7s. 9d. per chaldron?

$5 \times 5 = 25$

£	s.	d.	
2	7	9	
		5	
11	18	9	
		5	
59	13	9	*Ans.*

5. What must be paid for 105 hundred weight of bullets, at 5s. 7¾d. per hundred weight?

$3 \times 5 \times 7 = 105$

£	s.	d.	
	5	7¾	
		7	
1	19	6¼	
		5	
9	17	7¼	
		3	
£29	12	9¾	*Ans.*

85. If the multiplier cannot be produced by the multiplication of two or more single figures, take the nearest number to it which can be so produced, and multiply by its factors as before. Then augment, or diminish the result by as many times the multiplicand as the said number is less or greater than the multiplier.

6. At £4 1s. 10¼d. per thousand, what is the cost of 58 thousand bricks?

	£	s.	d.	
	4	1	10¼	
			8	
	32	14	10	
			7	
	229	3	10	price of 56.
	8	3	8½	price of 2 add.
Ans.	£237	7	6½	

Or thus:

	£	s.	d.	
	4	1	10¼	
			10	
	40	18	6½	
			6	
	245	11	3	price of 60.
	8	3	8½	price of 2 subtract.
Ans.	£237	7	6½	as before.

7. Multiply 2 2 10½ by 23.

	yd	f.	in.	
	2	2	10½	
			7	
	20	2	9	
			4	
	82	2	11 0	product by 28.
	2	2	10 ½	add.
Product	85		1¼ 3	

56. *Examples in Division by whole numbers.*

1. When oats are at £1 17s 9d per quarter, what is that per bushel?

	£	s	d	qr.	
8)	1	17	9	0	
	0	4	8	2⅜	*Ans*

2. If the interest of £100 for a year is £3⅞, what is the interest of £10 for that time?

£ 3⅞ = £3 7s 6d

	£	s	d.	
10)	3			
	0	6		the interest for 10.
	2	7	1	*Ans.*

57. When the divisor is the product of two or more simple numbers, divide by them separately. (30)

3. If a chaldron of coals cost £2 10s what is that per bushel?

6) 2 10 0

6) 0 8 4

0 1 4⅔ *Ans.*

4. If £3 guineas the hundred weight, what is that per lb.?

2 × 7 × 8 = 112

	£	s.	d.	
2)	3	3	0	
7)	1	11	6	
8)	0	4	6	
	0	0	6¾	*Ans.*

5. What is the 24*th* part of 19 2 9·5?

	y.	*f.*	*in.*
4)	19	2	9·5
6)	4	2	11·375
Ans.	0	2	5·89583 &c.

58. If the divisor cannot be resolved into small factors, divide by the whole at once after the manner of long division.

6. If the whole pay of 179 men for 61 days be 625 [illegible] what is the daily pay of each?

```
      £   s.  d.      £  s.  d
179 ) 625 11  4½  (  3   9  10¾  the whole pay of 1 man.
      537
       88
       20
     1771
     1611
      160
       12
     1924
      179
      134
        4
      537
      537
```

```
      £  s.  d      s. d
61 )  3  9  10¾  (  1  1¾.  Answer.
     60
     69
     61
      8
     12
    106
     61
     45
      4
    183
    183
```

59. III. When the multiplier, or the divisor, is a vulgar fraction, it is evident that the product in the former case, and the quotient in the latter, will each be obtained by both multiplication and division, except the numerator of the fraction be 1. — For the product of the numerator and multiplicand divided by the denominator will give the answer in multiplication. And the product of the denominator and dividend divided by the numerator is the quotient in division.

Examples.

Ton. C. lb.

1. If 86 17 100 of provisions will serve a garrison 12 months, what quantity will be necessary for 8 months?

8 months = $\frac{2}{3}$ of 12 months.

	T.	C.	lb.
	86	17	100
			2
3)	173	15	88
	57	18	66$\frac{2}{3}$ *Ans.*

£ s. d.

2. If I agree to give a labourer 1 6 6 for working 10 days, what must I pay him for 7 days?

Here 7 days is $\frac{7}{10}$ of the whole time.

	£	s.	d.	qr.
	1	6	6	0
			7	
10)	9	5	6	0
	0	18	6	2 *Ans.*

yds f. in.

3. What is $\frac{3}{5}$ of 79 8 34·7 square measure?

	y.	f.	in.
	79	8	34·7
			3
5)	239	7	30·1
	47	8	96·42

£ s. d.

4. If 7 hundred weight cost 6 13 4, what is that per ton?

cwt

Here 7 is $\frac{7}{20}$ of a ton, therefore $\frac{7}{20}$ is the divisor.

	£	s.	d.
	6	13	4
			2
	13	6	8
			10
7)	133	6	8 mult. product by 10.
	19	0	11$\frac{3}{7}$ *Ans.*

90. When the multiplier is a mixt number, the multiplication may be made by the parts separately and the products added together for the answer. If the divisor is a mixt number reduce it to an improper fraction. And when decimals are in the multiplier or divisor, reduce the multiplicand or dividend to the lowest denomination, and find the answer by the rules of Reduction.

OF ALIQUOT PARTS.

91. An *aliquot part* of a number is any other number which will divide it without leaving a remainder. Thus if the aliquot parts are confined to integers, 1, 2, and 3, will be all the aliquot parts of 6; 1 being the $\frac{1}{6}$, 2 the $\frac{1}{3}$, and 3 the $\frac{1}{2}$ of 6. Fractions and mixt numbers however, are aliquot parts, as $\frac{1}{7}$ or the 7th of 1 is an aliquot part of 1; $3\frac{1}{3}$ or $\frac{1}{3}$ of 10, an aliquot part of 10; $4\frac{1}{2}$ or $\frac{1}{3}$ of $13\frac{1}{2}$, an aliquot part of $13\frac{1}{2}$, &c. Also 3s. 4*d*. and 2s. 6*d*. are aliquot parts of a pound, the former being $\frac{1}{6}$, and the latter $\frac{1}{8}$. 4 inches is an aliquot part of a foot and also of a yard, being $\frac{1}{3}$ of the former, and $\frac{1}{9}$ of the latter, &c.

The principal use of aliquot parts is to abridge the operations in compound multiplication, or when several numbers of different denominations are to be multiplied together. The method by aliquot parts is also called *Practice*.

Examples.

1. What is the product of 74 by $6\frac{2}{3}$?

[illegible]

Ans. [illegible]

Here instead of multiplying by $\frac{2}{3}$ I take $\frac{1}{3}$ of the multiplicand, and again the $\frac{1}{3}$ of that [illegible], [illegible] both these parts together make $\frac{2}{3}$ of the multiplicand to be added to the product by 6.

2. Required the product of 782 and $20\frac{5}{8}$?

```
       782
       20⅝
     15640  ...... product by 20.
       391  ...... ½ or 4/8 of the multiplicand.
       97¾  ...... ¼ of 391, or ¼ of ½ or ⅛ of the multiplicand.
Ans. 16128¾
```

3. What will be the expense of a brick wall 785 yards long at 3s. 9d. per yard?

3s 9d. may be divided into two aliquot parts of a pound, *viz.* 2s. 6d. or ⅛, and 1s. 3d. or ½ *of* ⅛. And therefore it is evident that ⅛ *of* 785, and ½ *of* that ⅛ when added together will be the answer in pounds, &c.

```
s  d
2  6 = ⅛ ......... 8 ) 785
1  3 = ½ of ⅛ ... 2 )  98   2  6
                       49   1  3
                   £ 147   3  9
```

Or the aliquot parts may be taken as follows:

```
s.       £
2   = 1/10 ........... 10 ) 785
1s  = ½ ............... 2 )  78 10
6d  = ½ 1s ............ 2 )  39  5
3d  = ½ 6d ............ 2 )  19 12  6
                              9 16  3
                         £ 147  3  9  the answer as before.
```

4. If gunpowder is £5 13s 6d the hundred weight, what will 8 cwt. 1 qr 26 lb. cost?

```
 £   s.  d.
 5  13   6
          8
45   8   0  cost of 8 cwt.

                        £   s.  d.
1 qr = ¼ cwt ...... 4 ) 5  13   6
                        1   8   4½  ...... cost of 1 qr.
   = [illegible] ...... 0  16   3¼  ...... of 16 lb.
   = [illegible] ......     4   0¾  ...... of [illegible] lb.
                       45   8   0  ...... of 8 cwt
                     £ 47  16   7¾  Ans.
```

In the last example I find the price of 8 cwt. by compound multiplication

tion; and that of 1*qr.* 20*lb.* by the aliquot parts of a hundred weight. Thus 1*qr.* is $\frac{1}{4}$ of a hundred, and its price $\frac{1}{4}$ of £5 15*s.* 6*d.*—16*lb.* is $\frac{1}{7}$ of a hundred, therefore its price is $\frac{1}{7}$ of £5 15*s.* 6*d.* and the price of 4*lb.* (making up the 20*lb.*) is $\frac{1}{4}$ of that $\frac{1}{7}$.

5. If I pay 4*s.* 10$\frac{1}{4}$*d.* per yard, what will be the expense of 79 *yds.* 2 *f.* 6 *in.*?

	s.	*d.*	
1½ feet = ½ a yard	2	5$\frac{1}{8}$	expense of 2 *f.* 6 *in.* at 4*s.* 10$\frac{1}{4}$*d.* per *yd.*
1 foot = ⅓ of a yard ...	1	7$\frac{5}{12}$	

	79		
	4	10$\frac{1}{4}$	
shill.	316	0	.... 79 *yds.* at 4*s.*
6*d.* = ½ 1*s.*	39	6	expense of 79 *yds.* at 10$\frac{1}{4}$*d.*
3*d* = ½ of 6*d.*	19	9	
1*d.* = ⅓ of 3*d.*	6	7	
1 *far.* = ¼ of 1*d* ...	1	7$\frac{3}{4}$	
	2	5$\frac{1}{8}$	expense of 2 *f.* 6 *in.* at 4*s.* 10$\frac{1}{4}$*d.*
	1	7$\frac{5}{12}$	
shill.	387	6$\frac{7}{24}$	
£19	7	6$\frac{7}{24}$	*Ans.*

But the result may be obtained more concisely thus: Since 79 *yds.* 2 *f.* 6 *in.* is only 6 inches or $\frac{1}{6}$ of a yard short of 80 yards, if $\frac{1}{6}$ of 4*s.* 10$\frac{1}{4}$*d.* be deducted from the expense of 80 yards, the remainder will evidently be the answer required.

	£	*s.*	*d.*	
		4	10$\frac{1}{4}$	
			8	
	1	18	10	
			10	
	19	8	4	80 yards at 4*s.* 10$\frac{1}{4}$*d.*
$\frac{1}{6}$ of 4*s.* 10$\frac{1}{4}$*d.* = ...	0	0	9$\frac{17}{24}$	subtract
	£19	7	6$\frac{7}{24}$	*Ans.* as before

6. What is the product of 16 *f.* 7½ *in.* by 22 *f.* 10 *in.*?

	f.	*in.*		
	16	7½		
	22	10		
	352	0	= 16 × 22	
6 *in.* = ½ a foot	11	0	= ½ of 22	= 22 × 7½ *in.*
1½ *in.* = ¼ of 6 *in.*	2	9	= ¼ of 11	
6 *in.* = ½ a foot	8	3¾	= ½ of 16 7½	= 16 7½ × 10 *in.*
4 *in.* = ⅓ a foot	5	6½	= ⅓ of 16 7½	
Ans.	379	7¼		

This product is square measure, and therefore 379 are square feet, and the 12ths of a square foot, equal to $7\frac{3}{4} \times 12$ or 57 square inches.

Required the product of 45 f. $5\frac{1}{2}$ in. and 8 f. $9\frac{3}{4}$ in.

	f. in.	
	45 $5\frac{1}{2}$	
	8 $9\frac{3}{4}$	
	371 [illegible]	Product by 8.
6in. = $\frac{1}{2}$ a foot	[illegible] $2\frac{3}{4}$	= $\frac{1}{2}$ the multiplicand.
3in. = $\frac{1}{2}$ of 6in	11 [illegible]	= $\frac{1}{2}$ of [illegible]
$\frac{3}{4}$in. = $\frac{1}{4}$ of 3in.	[illegible]	= $\frac{1}{4}$ of 11 $7\frac{3}{8}$.
	409 [illegible]	or 409sq. f. and $59\frac{5}{8}$ sq. in.

3. Let [illegible] be multiplied by 10 yds. 1 f. 10 in.?

	yds. f. in.	
	[illegible]	
	[illegible]	
	[illegible]	product by 10.
[illegible]	[illegible]	the multiplicand.
[illegible]	[illegible]	= [illegible] of 10 1 11.
[illegible]	[illegible]	= [illegible] of 10 1 11.
	[illegible]	

Here [illegible] 238 will be square yards, [illegible] of a square yard, and [illegible] a square yard. And the whole [illegible]

[illegible] by aliquot parts will frequently be more expeditious than decimals for obtaining the contents of the [illegible] works, &c.

OF THE RULES OF PROPORTION.

I. Of Direct Proportion.

92. If 4 numbers are such that the first divided by the second is equal to the third divided by the fourth, or the second divided

by the first equal to the fourth divided by the third, they are said to be directly proportional.

Let the numbers be 2, 4, 5, 10: Then $\frac{2}{4} = \frac{5}{10}$; and $\frac{4}{2} = \frac{10}{5}$.

The fraction $\frac{2}{4}$ denotes the *ratio*, or rather the *exponent* of the *ratio* of 2 to 4, or of 5 to 10, because $\frac{5}{10} = \frac{2}{4}$: And $\frac{4}{2}$ the ratio of 4 to 2, or of 10 to 5.

The numbers or terms of the proportion are usually set down thus 2 : 4 :: 5 : 10, and read thus, as 2 is to 4, so is 5 to 10; which signifies that 2 bears the same proportion to 4, as 5 does to 10: This is evident, since 2 is the half of 4, and 5 is the half of 10; or 2 is contained in 4 the same number of times as 5 is contained in 10.

Hence if two fractions are equal, their terms are proportional.

For $\frac{2}{4} = \frac{5}{10}$; and 2 : 4 :: 5 to 10.

93. Since equal numbers multiplied by equal numbers must give equal products, if the equal fractions $\frac{2}{4}$, $\frac{5}{10}$ are multiplied by [illegible] or any other number, the products will be equal; namely [illegible], or (when the fraction [illegible] is changed) [illegible] = 10; therefore *the product of the* [illegible] *terms of the proportion*, 2 : 4 :: 5 : 10, *divided by the first term, gives the fourth term* 10. Consequently the product of the two middle terms 4 × 5, is equal to 2 × 10, the product of the other two.

94. Hence the rule of proportion is called the RULE OF THREE, because from *three given numbers* a *fourth* may be found, which shall have the same proportion to one of the three, as there is between the other two.

For example. If a body of troops in 2 hours march 4 miles; how far would they march in [illegible] hours at the same rate?

Here it is evident that the two distances will be in the same direct proportion as the times 2 and 5, or that 5 will have the same proportion to the required distance or 4th term, as 2 has to the distance 4.

Therefore having set down the three given terms or numbers in the order they are proposed, *multiply the 2d. and 3d together, and divide the product by the first, for the answer.*

$$\overset{h.}{2} : \overset{h.}{4} :: \overset{m.}{5} : \frac{\overset{m}{4 \times 5}}{2} = \overset{m.}{10}, \text{ [illegible]}$$

Or because $5 \times 4 = 4 \times 5$, the proportion may stand thus,

$$\overset{h.}{2} : \overset{m.}{[illegible]} :: \overset{h.}{[illegible]} : \frac{\overset{m.}{5 \times 4}}{2} = 10, \text{ as before}$$

The terms 2 and 4 are called the terms of *supposition*, and 5 that of *demand*: therefore in setting down the three given numbers of a proportion, or stating the question, always make that number the first term, which is of the same kind as the [illegible].

[illegible] fractions, [illegible], [illegible], are pro[illegible] lowest terms we have [illegible] 10; or 1 : [illegible] 2 [illegible] 10. [illegible] directly proportional, if the first [illegible] third terms are divided [illegible] term will still be the same.

[illegible] :: 5 : 10. And [illegible] 10.

[illegible] *multiples* or *sub-multiples* of [illegible] numbers, are [illegible] proportion as the numbers themselves.

[illegible] the operations in working proportions may sometimes be abridged, as in the following question:

If [illegible] of bread for a month, how many rations will [illegible] men require?

$$\text{As } \overset{m.}{900} : \overset{r.}{[illegible]} :: \overset{m.}{1170} : \overset{r.}{3[illegible]100} \text{ the } \textit{answer},$$

Or, dividing the two first terms by 500,

As 1 : 30 :: 1170 : 30 × 1170 = 35100 : where the product of the 2d. and 3d. terms is the 4th. term or answer.

97. Hence, if to several numbers we respectively add other numbers in the same proportion, the *sums* will also be in that same proportion. For the latter numbers may be considered as *like* multiples or sub-multiples of the former.

Thus, if to 3, 4, 6, we add 1, $1\frac{1}{3}$, 2 (having the same proportion) respectively; the sums will be 4, $5\frac{1}{3}$, 8, which are in the same proportion as 3, 4, 6. And the like is also evident with respect to the *differences*.

98. Hence also, it appears that fractions having a common denominator are in the same proportion as their numerators.

Thus the fractions $\frac{6}{12}$, $\frac{8}{12}$, $\frac{9}{12}$, are in the same proportion as 6, 8, 9. But the fractions when reduced to their lowest terms are [illegible], consequently those fractions are in the same proportion as 6, 8, 9.

99. If 4 numbers are directly proportional, then, as the sum of the 1st. and 2d. is to the 2d. (or 1st.) so is the sum of the 3d. and 4th. to the 4th. (or 3d.)

Suppose 2 : 4 :: 5 : 10.
Then 2 + 4 : 4 :: 5 + 10 : 10
And 2 + 4 : 2 :: 5 + 10 : 5.

For [illegible] equal numbers added to equal numbers [illegible] given [illegible] fraction (or 1) be added to $\frac{2}{4}$, [illegible] to [illegible] the sums must be equal,

viz. $\frac{2}{4}$ + [illegible] = [illegible], or $\frac{2 + 4}{4} = \frac{5 + 10}{10}$; these fractions being equal, their terms will be proportional,

Or 2 + 4 : 4 :: 5 + 10 : 10

In like manner [illegible] by adding [illegible] and [illegible] to [illegible] and [illegible] respectively, we have [illegible] (And if we subtract the equal fractions [illegible] instead of adding them, it may be proved that the *differences* are proportional).

For $2 + 4 : 5 + 10 :: 2 : 5$, therefore $\frac{2+4}{5+10} = \frac{2}{5}$. Now, if for $\frac{2}{5}$ we take any fraction equal to it, as $\frac{8}{20}$, we have $\frac{2+4}{5+10} = \frac{8}{20}$;

Therefore $2 + 4 : 5 + 10 :: 8 : 20$; or $2 + 4 : 8 :: 5 + 10 : 20$;

Hence, as $2 + 4 + 8 : 5 + 10 + 20 :: 8 : 20$;

or $2 + 4 + 8 : 5 + 10 + 20 :: 4 : 10$,

$:: 2 : 5$.

100. Hence is derived the method of dividing a number into a proposed number of parts having given proportions. Let 35 (or $5 + 10 + 20$) be divided into 3 parts which shall be as 2, 4, and 8. Then,

$2 + 4 + 8 : 35 :: 2 : 5$

$2 + 4 + 8 : 35 :: 4 : 10$ *Ans.* 5, 10, and 20.

$2 + 4 + 8 : 35 :: 8 : 20$

Again, suppose it be required to divide 100 into 3 parts having the proportions of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

The three fractions when brought to a common denominator are $\frac{6}{12}$, $\frac{4}{12}$, $\frac{3}{12}$, and therefore the parts will have the same proportions as the numbers 6, 4, and 3:

$$6 + 4 + 3 = 13,$$

Therefore $13 : 100 :: 6 : 46\frac{2}{13}$
$13 : 100 :: 4 : 30\frac{10}{13}$ } the 3 parts required
$13 : 100 :: 3 : 23\frac{1}{13}$

Again, for example, let $10\frac{5}{8}$ be divided into 2 parts having the same ratio as the decimals ·05 and ·075.

·05
·075
·125

Then $·125 : 10·625 :: ·05 : 4·25$
$·125 : 10·625 :: ·075 : 6·375$ } the two required parts.

II. Of *Inverse* or *Reciprocal Proportion*.

101. WHEN 4 numbers (2 : 4 :: 5 : 10) are in direct proportion, (as above) the product of the two middle terms (4×5) is equal to that of the other two (2×10). But if the proportion is inverse or reciprocal, the product of the two first terms will be equal to the product of the two last; or the ratio of the first term to the third is equal to that of the fourth to the second.

For example: If 4 men can do a piece of work in 6 days, in what time would 8 men do the same?

Taking the proportion direct, the answer comes out 12 days; but the true time is evidently no more than 3 days, because 8 men will require but half the time which 4 require.

$$\begin{array}{llll} m. & d. & m. & d. \end{array}$$
As 4 . 6 :: 8 : 3. Here $\frac{4 \times 6}{8} = 3$; [illegible] the product of the [illegible] first terms divided by the [illegible] term [illegible] hence this

Rule. [illegible] the term of supposition [illegible] the [illegible] and divide the product by the term of demand for the [illegible] answer.

[illegible] example; suppose 40 men [illegible] a yard is allowed [illegible] each rank they [illegible] But if the same 40 stand 4 in a rank, [illegible] (allowing [illegible] to each rank as before), [illegible] that the lengths [illegible] inversely [illegible] of men in front.

$$\begin{array}{llll} m. & yds. & m. & yds. \end{array}$$
[illegible] 5 : 8 :: 4 : 10. [illegible] $\frac{5 \times 8}{4} = 10$. And the ratio of 5 to 4 is equal to that of 10 to 8; or [illegible]. The terms of supposition being 5 and 8, and that of demand 4,

102. To discover when a proportion should be wrought inversely, consider if *more* requires *less*, or *less* requires *more*, or if one number *increases* in the same proportion as another *diminishes*, for in either case the inverse rule must be used.

103. *N. B.* When the two terms of a proportion which are of the same kind, are given in different denominations, reduce them to the same denomination. Thus if one is pounds, &c. and the other pence, &c. reduce them both to pounds, or to pence. If one is feet and inches, and the other inches, reduce them both to feet or to inches, &c. And the fourth term or answer will always be in that denomination to which the given term of the same kind is reduced.

Questions in *Compound Proportion* or the *Double Rule of Three*, may always be answered by two or more single statings.

Examples.

1. To find a [illegible] proportional [illegible] and 3[illegible]

[illegible] : 3[illegible] : [illegible] 5 *Ans.*

[illegible]

) [illegible]-9

[illegible]

[illegible] by reversing the question.—Thus to find a [illegible] proportional [illegible]

As 1[illegible] : [illegible] :: 33 : $\frac{33 \times [illegible]3}{217\cdot 8}$ = 5 the *Ans.*

2. To find a 4th proportional to 11·5, ·0769 and 1000

As [illegible] : ·0769 :: 1000 : 6·6869 &c. *Ans.*

[illegible]

[illegible]) [illegible]000 (6·6869 &c

[illegible]

[illegible]90

[illegible]

1[illegible]

[illegible]

[illegible]00

[illegible]

3. Let a *4th.* proportional to the three fractions [illegible], [illegible], and $\frac{5}{21}$, be required.

As [illegible] : [illegible] :: $\frac{5}{21}$: [illegible] $\times$ [illegible] $\times$ $\frac{5}{21}$ = [illegible] $\times \frac{1}{3} \times$ [illegible] = [illegible] the *Ans*

N. B. It will be advisable in most cases to set down the *4th.* term in the form of a vulgar fraction, and then reduce it to its lowest terms, as in the last example.

4. What are coals per chaldron when three bushels cost 4 shillings?

36 *bush.* = 1 chaldron. Therefore we have to find a *4th.* proportional to 3, 4, and 36:

$$\overset{bush.}{\text{As } 3} : \overset{shill.}{4} :: \overset{bush.}{36} : \overset{shill.}{\frac{4 \times 36}{3}} = 48 \textit{ shill.} \text{ the } \textit{Ans.}$$

5. The quick time, or step in marching being 2 paces *per* second or 120 *per* minute at 2½ feet each; at what rate *per* hour does a troop march? and what time is taken up in marching 6 miles?

$120 \times 2\frac{1}{2} \times 60 = 18000$ feet *per* hour = [illegible] miles.

As [illegible] : [illegible] :: 6 : [illegible] = [illegible] = [illegible] min. = [illegible]

Ans [illegible] *m. per* hour [illegible]

6. What will the tax on £514 15s. be at 1s. 6d. in the pound?

As £1 : 1s. 6d. :: £514 15s. : [illegible]

Or as 240 d. : 20 d. :: £514 15s. : [illegible]

Or (90) dividing the two first terms by 20, we have

As 12 : 1 :: £514 15s. : $\frac{514\ 15}{12}$ = £42 17s. 11d. *Ans.*

7. What is the rent *per ann.* of 140 ac. 3 roods 20 pol. at £1 10 8 *per* acre?

ac. r. p.
140 3 20 = 22540 poles.

£ s. d.
1 10 8 = 368 pence.

As $\overset{p.}{160} : \overset{d.}{368} :: \overset{p.}{22540} : \overset{d.}{\frac{368 \times 22540}{160}} = \overset{d.}{51842} = \overset{\pounds\ \ s.\ \ d.}{216\ 0\ 2}$ the answer.

8. A sets out from Oxford to London at the same time that B leaves London for Oxford, the former travels 5, and the latter 6 miles an hour; now supposing Oxford to be 58 miles from London, how far from the latter place will they meet if they travel the same road?

If the whole distance be divided into two parts having the proportion of 5 to 6, it is evident those parts will be the respective distances travelled.

(100) As $5 + 6 : 58 :: 6 : 31\frac{7}{11}$ m. travelled by B.
$5 + 6 : 58 :: 5 : 26\frac{4}{11}$ travelled by A.

9. A detachment sets out at 6 in the morning, marching at the rate of $1\frac{3}{4}$ miles an hour; 3 hours after, another detachment from the same place follows them, but their march is $2\frac{1}{2}$ miles an hour. In what time will the latter overtake the former; and what distance will they have marched?

$1\frac{3}{4} \times 3 = 5\frac{1}{4}$ m. the first detachment is a-head when the other begins its march.

The difference of $2\frac{1}{2}$ m. and $1\frac{3}{4}$ m. is $\frac{3}{4}$ m., what the latter gains on the former per hour.

But it has to gain $5\frac{1}{4}$ m. in the whole.

Therefore, as $\overset{m.}{\frac{3}{4}} : \overset{h.}{1} :: \overset{m.}{5\frac{1}{4}}$
or, as $\frac{3}{4} : 1 :: \frac{21}{4}$

(98), or, as $3 : 1 :: 21 : \frac{21}{3} = 7^{h.}$ the *time required.*

And $2\frac{1}{2} \times 7 = 17\frac{1}{2}$ miles the *distance required.*

10. The hour and minute hands of a watch are together at 12 o'clock; at what time are they next together?

The minute hand moves 1 circumference on the dial plate in 1 hour;
but the hour hand moves only $\frac{1}{12}$
the difference is $\frac{11}{12}$ which the minute hand gains per hour

But at setting off at 12 o'clock we may consider the hour hand as being 1 circumference before the minute hand;
Therefore the minute hand has to gain 1 circumference

As $\frac{11}{12} : 1h. :: 1 : \frac{12}{11}h. = 1\frac{1}{11}h.$, the *answer.*

11. There is an Island 29 miles in circumference, and three travellers all start together to travel the same way about it, A goes 3 miles per hour, B 5, and C 7; when will they all be together again?

B gains 2 miles an hour upon A;

Therefore as $2^{m.} : 1^{h.} :: 29^{m.} : 14\frac{1}{2}^{h.}$ the time from starting when B overtakes A.

C gains 4 miles an hour upon A,

Hence $4^{m.} : 1^{h.} :: 29^{m.} : 7\frac{1}{4}^{h.}$ the time when C overtakes A

And since C will overtake A at the end of every $7\frac{1}{4}$ hours, they will be together at the end of twice $7\frac{1}{4}$ hours, or $14\frac{1}{2}$ hours

Therefore all three will be together again at the end of $14\frac{1}{2}$ hours from the time of starting

12. Suppose a clock has 4 hands, A, B, C, D; and that A goes round once in 5*d.* 20*h.* B in 7*d* 11*h* C in 10*d.* 20*h.* and D in 18*d* 23*h.* Now if the hands are all together at any particular time, how long will it be before they come in conjunction again?

d.	h.	h.
5	20 =	140
7	14 =	182
10	20 =	260
18	23 =	455

the times of revolution.

Now it is evident that at the end of any number of hours which is a common multiple of 140, 182, 260, and 455 (the times of revolution) the hands will be together again: but the least common multiple is 13 × 5 × 7 × 4 or 1820 (46); therefore in 1820 hours,

A will have moved 13 }
B .... 10 } times round,
C .... 7 }
D .... 4 }

Consequently at the end of every 1820 hours the hands are together at the *same place*.

Therefore since the hands come together at every like whole multiple of 13, 10, 7, 4 revolutions (as twice, thrice, four times, &c.), it follows that if we can find like sub-multiples or aliquot parts of 13, 10, 7, and 4, *having like fractions*, the hands must have been in conjunction without performing entire revolutions. Thus, if we divide 13, 10, 7, and 4 by 3, we get $4\frac{1}{3}$, $3\frac{1}{3}$, $2\frac{1}{3}$, $1\frac{1}{3}$ revolutions for the elapsed time, or $\frac{1820}{3} = 606\frac{2}{3}$ *hours* the time required.

Or thus.

A moves $\frac{1}{140}$, B $\frac{1}{182}$, C $\frac{1}{260}$, and D $\frac{1}{455}$ of the circumference in 1 hour, respectively.

Now if we proceed according to *Examp.* 7.
we have $\frac{1}{140} - \frac{1}{455} = \frac{9}{1820}$ of the circumference which A gains on D in 1 hour:
Therefore $\frac{9}{1820}$ *circumf.* : 1*h.* :: 1 *circumf.* : $202\frac{2}{9}$*h.* the time in which A is overtaking D.

And $\frac{1}{182} - \frac{1}{455} = \frac{3}{910}$ circumf. which B gains on D in 1 hour:
As $\frac{3}{910}$: 1*h.* :: 1 : $303\frac{1}{3}$*h.* the time in which B is overtaking D.

Also $\frac{1}{260} - \frac{1}{455} = \frac{3}{1820}$ circumf. which C gains on D in 1 hour;
And $\frac{3}{1820}$: 1*h.* :: 1 : $606\frac{2}{3}$*h*, the time in which C is overtaking D.

Now it is evident that the least common multiple of $202\frac{2}{9}$, $303\frac{1}{3}$, and $606\frac{2}{3}$ will be the time when A, B, and C will first overtake D together; but $606\frac{2}{3}$ is the least common multiple; for twice $303\frac{1}{3}$ is $606\frac{2}{3}$, and three times $202\frac{2}{9}$ is $606\frac{2}{3}$; therefore $606\frac{2}{3}$ *hours* is the time, as before.

13. What length must be cut off a rectangular board that is $7\frac{1}{2}$ inches broad, to make a foot or 144 square inches?

In other words——What number is that which multiplied by $7\frac{1}{2}$ shall make 12 times 12, or 144?

Here the proportion will be inverse;

$$\begin{array}{lllllll} & \textit{broad} & \textit{long} & \textit{broad} & & & \\ \text{As} & 12 & : 12 & :: 7\frac{1}{2} & : & \dfrac{12 \times 12}{7\frac{1}{2}} & = 19\frac{1}{5} \text{ inches, } \textit{answer.} \end{array}$$

14. A garrison of 488 men have provisions for 39 weeks, how long will those provisions last if the garrison be increased to 732 men?

It is evident that the provisions will last a less time, therefore the proportion must be wrought inversely:

$$\begin{array}{llllll} & m. & w. & m. & & \\ \text{As} & 488 & : 39 & : 732 : & \dfrac{488 \times 39}{732} & = 26 \text{ weeks, } \textit{answer.} \end{array}$$

15. If 1000 men besieged in a town with provisions for 28 days, allowing 18 ounces a day per man, be reinforced with 600 men, and supposing that they cannot be relieved till the end of 42 days; how many ounces a day must each man have that the provisions may last that time.

$1000 \times 18 \times 28$ ounces, the whole quantity of provisions. This quantity is to last 1600 men 42 days.

Divide by 1600, and we have $\dfrac{1000 \times 18 \times 28}{1600}$ *ounces* the quantity which must last 1 man 42 days: this divided by 42 will give the allowance *per day* for 1 man, viz.

$$\frac{1000 \times 18 \times 28}{1600 \times 42} = \frac{10 \times 18 \times 28}{16 \times 42} = \frac{10 \times 9 \times 2}{8 \times 3} = \frac{5 \times 3}{2} = 7\tfrac{1}{2} \text{ oz.}$$

the *answer*.

16. If the carriage of 11 cwt. 3 qr. of baggage amounts to £5. 16 s. for 40 miles; what will be the expence of ton 17 cwt. for 94 miles at the same rate?

ton cwt. £ s.
6 17 = 548 qrs. 5 16 = 116 sh.

cwt. qrs.
71 3 = 287 qrs.

As 287 qrs. : 116 sh. :: 548 qrs. : $\frac{548 \times 116}{287}$ sh. the expence of 6 ton 17 cwt. for 40 miles.

As 40 m. : $\frac{548 \times 116}{287}$ sh. :: 94 m. : $\frac{548 \times 116 \times 94}{287 \times 40}$ sh. $= \frac{137 \times 116 \times 47}{287 \times 5} =$

$520\frac{724}{1435}$ sh $=$ £26 0s. $6\frac{78}{1435}$d. the *answer*.

17. If a company of 160 men in six days of 11 hours each, can dig a trench 230 yards long, $5\frac{1}{2}$ wide, and $1\frac{1}{2}$ deep; in how many days of 8 hours long would another company consisting of 96 men dig a trench 220 yards long, $3\frac{1}{2}$ wide, and 1 deep; supposing the hardness of the ground in the former case is to that in the latter as 5 to 7, and that 4 men of the latter company can do as much work as 5 of the former in the same time?

$230 \times 5\frac{1}{2} \times 1\frac{1}{2} = 1897\frac{1}{2}$ (by mensuration) the cubic yards in the first trench.

$220 \times 3\frac{1}{2} \times 1 = 770$ the cubic yards in the other.

Now if we suppose the labour necessary to raise a like quantity of earth to be directly proportional to the hardness of the ground, it is evident that the strength required to dig the former trench, will be to that required for the latter, as $1897\frac{1}{2}$ yds. $\times 5$ to 770 yds. $\times 7$.

And, as 4 m. : 5 m. :: 96 m. : 120 m., therefore the labour of 120 men of the first company is equal to that of the 96 men.

Hence the question is reduced to the following.

If 160 men in 66 hours (6×11) can dig $1897\frac{1}{2}$ yds. $\times 5$; in what time would 120 men dig 770 yds. $\times 7$?

As 160 m. : $1897\frac{1}{2}$ yds. $\times 5$ :: 120 m. : $\frac{1897\frac{1}{2} \times 5 \times 120}{160}$ yds., the yards which 120 men could dig in 66 hours.

As $\frac{\overset{yds.}{1897\frac{1}{2}} \times 5 \times 120}{160} : \overset{h.}{66} :: \overset{yds.}{770} \times 7 : \frac{66 \times 770 \times 7 \times 160}{1897\frac{1}{2} \times 5 \times 120} = \frac{23 \times 154 \times 7 \times 8}{3795}$ *hours*, which divided by 8 gives $6\frac{2446}{3795}$ days the *answer*.

Questions of this kind however, may be answered in the following manner: Set down the several proportions in succession, remembering to make the term of supposition which is of the same kind as the required answer, the second term of each proportion; then if the proportions are compounded (140) it will be reduced to a single stating.

Thus, the required answer being *days*, 6 days will be the second or middle term.

	d.		
As 160*m.*	6	:: 96*m.*	*(inverse).*
11*h.* :		:: 8*h*	*(inverse)*
230*l.* :		:: 220*l.*	
$5\frac{1}{2}$*br.* :		:: $3\frac{1}{2}$*br.*	
$1\frac{1}{2}$*d.* :		:: 1*d*	
5*har.* :		:: 7*har.*	
5*m.* :		:: 4*m.*	

And the divisors are 96, 8, 230, $5\frac{1}{2}$, $1\frac{1}{2}$, 5, and 5, therefore

As $96 \times 8 \times 230 \times 5\frac{1}{2} \times 1\frac{1}{2} \times 5 \times 5 : \overset{d}{6} :: 160 \times 11 \times 220 \times 3\frac{1}{2} \times 1 \times 7 \times 4 : \frac{6 \times 160 \times 11 \times 220 \times 3\frac{1}{2} \times 1 \times 7 \times 4}{96 \times 8 \times 230 \times 5\frac{1}{2} \times 1\frac{1}{2} \times 5 \times 5}$ *days*, which reduced to its lowest terms is $\frac{25216}{3795}$ *days*, the *answer* as before.

The three last questions and others of the same kind, belong to what is usually denominated the *Double Rule of Three.*

18. A detachment consisting of 4 companies being sent into a garrison in which the duty requires 60 men a day; what number must each company furnish in proportion to its strength; the first consisting of 42 men, the second of 49, the third of 56, and the fourth of 63?

It is evident that 60 must be divided into 4 numbers having the proportions of 42, 49, 56, and 63.

	42
	49
	56
	63
Sum	210

	men		men	
	As 210 : 60 :: 42 :		12	from the 1st. company.
	210 : 60 :: 49 :		14	2d.
(100)	210 : 60 :: 56 :		16	3d.
	210 : 60 :: 63 :		18	4th.

19. Two troops of horse rent a field for which they pay £82: one troop sent 64 horses for 25 days, and the other sent 56 horses for 30 days. How much of the rent must each troop pay?

Suppose 1 is the quantity of grass which a horse eats in 1 day:

Then 64 horses will eat 64×25 (1600) such quantities in 25 days.

And 56 horses will eat 56×30 (1680) such quantities in 30 days.

Now it is evident that the shares of the rent will be in the same direct proportion as the quantities consumed, or as 1600 and 1680. Hence the following rule for questions of this kind:

Multiply each stock by the time of its continuance, then divide the whole quantity to be parted into shares in the same proportion as those products.

(100) 1600
1680
3280

As 3280 : £82 :: 1600 : £40 what one troop must pay.
3280 : 82 :: 1680 : 42 what the other must pay.

The last question, and others of the same kind, belong to the rule called *Double Fellowship.*

20. To divide 108 into three such parts, that $\frac{1}{2}$ the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third may be equal each other.

Assume 3 numbers which shall be in the same proportion as the required parts.

Suppose $\left\{\begin{matrix}2\\3\\4\end{matrix}\right\}$ where the $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ are equal.

Then (100)

$$\left.\begin{array}{l} \text{As } 2+3+4\ (9) : 108 :: 2 : 24 \\ \qquad\qquad\quad 9 : 108 :: 3 : 36 \\ \qquad\qquad\quad 9 : 108 :: 4 : 48 \end{array}\right\}\ \text{the three parts required.}$$

91. A general after detaching $\frac{4}{11}$ of his army to occupy a certain height, and $\frac{7}{11}$ of the remainder to watch the enemy's motions, had only 700 men left. Query the whole number of troops?

If we suppose the army to be 1, then $\frac{7}{11}$ will be left when $\frac{4}{11}$ is detached.

And $\frac{7}{11}$ *of* $\frac{7}{11}$ or $\frac{49}{121}$ will be the strength of the *2d.* detachment.

And $\frac{4}{11} + \frac{49}{121} = \frac{93}{121}$ will be both detachments; this taken from 1, and $\frac{28}{121}$ of the army remains, which by the questions is equal to 700:

Therefore as $\frac{28}{121} : 700 :: 1 : \frac{121 \times 700}{28} = 3025$ the number required.

Questions which can be answered in a manner similar to the two last, are generally classed under the Rule of *Single Position.*

22. Sold a horse for 40 guineas, by which I lost 4 *per cent.* whereas in dealing I ought to have gained 10 *per cent.* How much was it sold for under its value?

$$\begin{array}{rl} \pounds & \\ 100 & \\ 4 & \text{subtract} \\ \hline 96 & \end{array}$$

$$\text{As } \overset{\pounds}{96} : \overset{\pounds}{100} :: \overset{G}{40} : \overset{G.}{\frac{100 \times 40}{96}} \text{ the prime cost}$$

$$\text{And, as } \overset{\pounds}{100} : \overset{\pounds}{110} :: \overset{G.}{\frac{100 \times 40}{96}} : \overset{G.}{\frac{110 \times 100 \times 40}{100 \times 96}} = 45\tfrac{5}{6} \text{ guineas its}$$

price at 10 *per cent* profit.

$$\begin{array}{rl} 45\frac{5}{6} & \\ 40 & \text{sub.} \\ \hline 5\frac{5}{6} & \text{guineas, the } \textit{answer.} \end{array}$$

23. Suppose on a march, a party of foot is 1000 paces before another of horse, and the rate of marching is 6 paces by the foot to 5 by the horse; now if two horse's steps be equal to $2\frac{1}{2}$ of a man's, how many paces will the horse take to come up with the foot?

Because 1 horse's pace is equal to $1\frac{1}{4}$ man's paces, 5 paces of a horse will be equal to $6\frac{1}{4}$ man's paces.

Therefore the horse at every 5 paces gains $\frac{1}{4}$ of a man's pace: and at this rate the party of horse have to gain 1000 man's paces;

$$\begin{array}{cccccccc} & m.p. & & h.p. & & m.p. & & h.\ paces. \\ \text{Hence, as} & \frac{1}{4} & : & 5 & :: & 1000 & : & 20000, \text{ the } answer. \end{array}$$

24. A can do a piece of work in 7 days, and B can do the like in 5 days; in what time would it be done if they work together?

$$\begin{array}{cccccccc} & d. & & w. & & d. & & w. \\ \text{As} & 7 & : & 1 & :: & 5 & : & \frac{5}{7} \text{ what A can do in 5 days.} \end{array}$$

Therefore both together can do $1\frac{5}{7}$ in 5 days.

$$\begin{array}{cccccccc} & w. & & d. & & w. & & d. \\ \text{As} & 1\frac{5}{7} & : & 5 & :: & 1 & : & \frac{35}{12} = 2\frac{11}{12} \text{ days, the } answer. \end{array}$$

25. A and B can perform a piece of work in 2 days; A and C in 3 days; and B and C in 5 days. in what time would each do it by himself?

$$\begin{array}{cccccccc} & d. & & w. & & d. & & w. \\ \text{As} & 3 & : & 1 & :: & 2 & : & \frac{2}{3} \text{ what A and C can do in 2 days.} \\ \text{As} & 5 & : & 1 & :: & 2 & : & \frac{2}{5} \text{ what B and C can do in 2 days.} \end{array}$$

$$\begin{array}{lll} & d. & w. \\ \text{By A and B in} & 2 \ldots\ldots & 1 \\ \text{By A and C in} & 2 \ldots\ldots & \frac{2}{3} \\ \text{By B and C in} & 2 \ldots\ldots & \frac{2}{5} \end{array}$$

Sum $2\frac{1}{15}$; but in doing this, each of the three must evidently work 4 days, therefore the three together would do half of $2\frac{1}{15}$ or $1\frac{1}{30}$ in 2 days.

$$\text{Hence } \left.\begin{array}{l} 1\frac{1}{30} - 1 = \frac{1}{30} \text{ what C} \\ 1\frac{1}{30} - \frac{2}{3} = \frac{11}{30} \text{ what B} \\ 1\frac{1}{30} - \frac{2}{5} = \frac{19}{30} \text{ what A} \end{array}\right\} \text{can do in 2 days.}$$

Therefore, as $\frac{1}{30}$: 2 :: 1 : 60 days the *time* by C.
$\frac{11}{30}$: 2 :: 1 : $5\frac{5}{11}$ .... by B.
$\frac{17}{30}$: 2 :: 1 : $3\frac{9}{17}$ .... by A.

26. The plan of a fortified town and its environs in the Netherlands is 15 inches long and 12 broad. The *scale* annexed to it is 800 toises, and is 4·7 inches in length. Now if the plan be enlarged to a scale of 6 inches the English mile; what will be the length and breadth?

A toise = 2·1315 yards.
2·1315 × 800 = 1705·2 yards the scale.

$$\text{As } \overset{yds.}{1705{\cdot}2} : \overset{in.}{4{\cdot}7} :: \overset{yds.}{1760} : \frac{4{\cdot}7 \times 1760}{1705{\cdot}2}$$ the length of a mile on the scale of toises.

And since the dimensions will be in the same proportion as the respective scales, we have,

$$\text{As } \overset{in.}{\frac{4{\cdot}7 \times 1760}{1705{\cdot}2}} : \overset{in.}{6} :: \overset{in.}{15} : \overset{in.}{\frac{1705{\cdot}2 \times 6 \times 15}{4{\cdot}7 \times 1760}} = \overset{in.}{18{\cdot}55}$$ the *required length.*

$$\text{And } \frac{4{\cdot}7 \times 1760}{1705{\cdot}2} : 6 :: 12 : \frac{1705{\cdot}2 \times 6 \times 12}{4{\cdot}7 \times 1760} = 11{\cdot}81$$ *inch the breadth.*

27. In what time would 16 battalions of infantry each consisting of 510 men, with two field pieces, 4 horses to each, pass through a defile $1\frac{1}{4}$ miles long, supposing the march is in open column with 6 men in front, and the rate 75 paces (of $2\frac{1}{2}$ feet each) *per minute*, being that of ordinary time?

Suppose a battalion in line, of 3 ranks; then $\frac{510}{3}$ = 170 men in each rank; and 22 inches or $1\frac{5}{6}$ feet being the allowance for each man in front, we have 170 × $1\frac{5}{6}$ = $311\frac{2}{3}$ feet the extent of the front or line, which also is the estimated extent of the same battalion when in open column.

$$\begin{array}{ll} & 311\tfrac{2}{3} \\ & 160 \quad \textit{feet}, \text{ extent of 2 field pieces with 4 horses to each.} \\ \hline \text{Sum} & 471\tfrac{2}{3} \quad \textit{feet}, \text{ extent of 1 battalion with 2 field pieces.} \end{array}$$

And 471[illegible] × 16 = [illegible] *feet*, extent of the 16 battalions, equal to 3019 *paces* of $2\frac{1}{2}$ *feet* each.

$$\begin{array}{rl} 3019 & \\ 3696 & \textit{paces} = 1\tfrac{3}{4} \text{ miles.} \\ \hline 6715 & \textit{paces}, \text{ extent of column and defile.} \end{array}$$

As 75 *pa.* : 1 *min.* :: 6715 *pa.* : $89\frac{8}{15}$ *min.* *Ans.*

28. Suppose 18 battalions each consisting of 560 men, with 18 mounted officers, and 2 field pieces (each with 4 horses) have to pass two defiles, one in a bad road, 1 mile in length; the other a good road, $1\frac{1}{2}$ miles long; each defile admitting of 3 men to march in front; how many battalions must pass each defile that the whole march through them may be made in the least time,

allowing { 6 feet in front to each rank of foot,
12 feet to a rank of horse;
80 feet for the extent of a field piece with 4 horses;
$2\frac{1}{2}$ feet the pace of a man:

And that { 80 paces *per minute* in a good road.
infantry march { 50 . . . . in a bad road.

In order that the whole march may be made in the least time, it will [illegible] to divide the 18 battalions into two columns whose lengths [illegible] that their rears may quit the defiles at the same time; or that the march of one column through one defile must be made in the same time as that of the other column through the other defile. This will evidently be when the length of one column added to a mile, is to the length of the other column added to $1\frac{1}{2}$ miles, as 50 to 80, the rates of marching in the defiles.

$$\begin{array}{rl} 3\,)\,561 & \\ 187 & \text{ranks} \\ 6 & \\ \hline 1122 & \text{extent of 187 ranks.} \\ 160 & \text{for 2 field pieces.} \\ 108 & \text{for 9 ranks of officers riding two and two} \\ \hline 1390 & \textit{feet}, \text{ extent of 1 battalion} = 556 \textit{ paces} \text{ of } 2\tfrac{1}{2} \textit{ feet} \text{ each.} \end{array}$$

556 × 18 = 10008 *paces*, extent of 18 battalions.
$\quad\;$ 2112 *paces* = 1 mile.
$\quad\;$ <u>3168</u> *paces* = $1\frac{1}{2}$ miles.
$\quad$ 15288 *paces*, extent of both columns and defilés.

80 + 50 = 130.

(100) As 130 : 15288 :: 80 : 9408 *paces*, length of the $1\frac{1}{2}$ mile defilé with its column.

And

As 130 : 15288 :: 50 : 5880 *paces*, the length of the 1 mile defilé with its column.

$\quad$ 9408
$\quad$ <u>3168</u> *paces* = $1\frac{1}{2}$ miles.
$\quad$ 6240 *paces*, length of the column which must pass the longest defilé; this divided by 556 the length of 1 battalion, gives 11 (the nearest whole number) for the number of battalions which must march through the $1\frac{1}{2}$ mile defilé.

$\quad$ 5880
$\quad$ <u>2112</u> *paces* = 1 mile.
$\quad$ 3768 *paces*, length of the column to pass the 1 mile defilé.

And $\frac{3768}{556}$ = 7 (the nearest integer) for the number of battalions which must march through the shortest defilé.

29. Suppose the same 18 battalions have to pass two defilés of equal extent, one admitting of 3, the other of 4 men in front; how must the 18 battalions be divided that the whole march through them may be performed in the least time, if the roads are equally good?

Since the rate of marching in each defile is the same, the extent of the columns must be equal. And therefore 18, the number of battalions, must be divided into two parts having the same proportion as the length of a battalion marching 3 men in front, to the length when 4 men march in front.

$\quad$ 443 *paces*, extent of a battalion 4 men in front.
$\quad$ <u>556</u> ... extent, 3 men in front (*see the last question*).
$\quad$ 999

As 999 : 18 :: 443 : 8 nearly.
$\quad$ 999 : 18 :: 556 : 10 nearly.

Therefore 10 battalions must march through the widest defilé, and 8 through the other.

30. To divide 20 into 2 such parts that the product of the first part by 5, shall be to the product of the other part by 6, in the proportion of 10 to 3?

It is evident that the two required parts will be in the same proportion as $\frac{10}{5}$ and $\frac{3}{6}$, because if the former of those fractions is multiplied by 5, and the latter by 6, the products will be in the given proportion; therefore 20 must be divided into two parts having the proportion of $\frac{10}{5}$ and $\frac{3}{6}$.

Hence (160) as $\frac{10}{5} + \frac{3}{6} : 20 :: \frac{10}{5} : 16$ the first part; consequently 4 is the other part.

In like manner any other number may be divided into a proposed number of parts such, that their products by given numbers may obtain given proportions.

31. Suppose 8 battalions have to pass 2 defilés, one $\frac{3}{4}$, the other $1\frac{3}{4}$ miles in length; the former admitting 6, and the latter 4 men to march in front; now if the length of a battalion including 2 field pieces be 330 *paces* of $2\frac{1}{2}$ *feet* each, when 6 men march in front, and 440 when 4 men march in front; how many battalions must pass each defilé that the whole march through them may be made in the least time, supposing the rate of marching in the shortest defilé is 50, and in the other 80 *paces per* minute?

It follows from *examp.* 28, that the length of one column added to $1\frac{3}{4}$ miles must be to the length of the other column added to $\frac{3}{4}$ mile, in the proportion of 80 to 50, the rates of marching.

$1\frac{3}{4}$ *m.* = 3696 *paces.*
$\frac{3}{4}$ *m.* = 1584 *paces.*

As 80 : 50 :: 3696 : 2310 *paces.*
1584 *paces* = $\frac{3}{4}$*m.*
726 *paces* dif. and $\frac{726}{330} = 2\frac{66}{330}$ battal.

Therefore if the extent of $2\frac{66}{330}$ battal. (726 *paces*) be added to the shortest defilé, the sum will be to the longest defilé, in the proportion of 50 to 80 the rates of marching. Consequently $5\frac{264}{330}$ battal (the difference of 8 and $2\frac{66}{330}$) must be divided into 2 such parts that the product of one

part by 330 shall be to the product of the other part by 440, in the proportion of 50 to 80.

Hence (by the last example):

As $\frac{50}{330} + \frac{80}{440}$: [illegible] :: $\frac{80}{440}$: 3 (the nearest integer) for one of the parts required; ([illegible]) which part is the number of battalions that must march through the longest defilé; consequently 5 have to march through the other.

$\frac{1584 + 330 \times 5}{50} = 64\frac{34}{50}$ *min.* the time of marching through the shortest defilé.

$\frac{3696 + 440 \times 3}{80} = 62\frac{56}{80}$ *min.* time of marching through the longest.

N. B. In this and the 28*th.* and 29*th.* examples it is supposed that the fronts of the columns enter the defilés nearly at the same time.

32. Suppose 40*lb.* of gunpowder at 1*s.* *per lb.* be mixt with 60*lb.* at 1*s.* 3*d.* *per lb.* what is 20*lb.* of the mixture worth?

40*lb.*	at 1*s.* .... is	40*s.*	
60	at 1*s.* 3*d.* .... is	75*s.*	
100		115*s.*	Therefore the value of 100*lb.* is 115*s.*

Hence, as 100*lb.* : 115*s.* :: 20*lb.* : 23*s.* the *answer*.

33. If the strength or quality of three sorts of gunpowder (or other ingredients) be denoted by 10, 13, and 16; how much of each must be taken that the proportionate quality of the mixture may be 12?

Or, putting the question in more familiar terms: Suppose 10, 13, and 16 pence, are the prices *per* pound; what quantity of each will make a mixture worth 12 pence *per* pound?

Because every *lb.* at 10*d.* gives 2*d.* less, and every *lb.* at 13*d.* gives 1*d.* *more* than 12*d.* the mean price, therefore 3*lb.* at 10*d.* to 2*lb.* at 13*d.* will make the defect below 12*d.* equal to the excess above it.

Thus 3*lb.* at 10*d.* will give 6*d.* less than 3*lb.* at 12*d.*
And 2*lb.* at 13*d.* will give 6*d.* more than 2*lb.* at 12*d.*

Hence the quantities will be reciprocally as the differences between the mean and extreme prices.

Therefore *3lb.* at 10*d.* and *2lb.* at 15*d.* will together be worth 12*d.* per *lb.*

Again, the difference of 10*d.* and 12*d.* is 2*d.*
and that of 12*d.* and 16*d.* is 4*d.*
Therefore *4lb.* at 10*d.* and *2lb* at 16*d* will together be worth 12*d.* per *lb.*

Consequently *7lb.* (4 + 3) at 10*d.* }
2 .... at 15 }
2 .... at 16 }
will be worth 12*d.* per *lb.* Or any quantities in the same proportion as 7, 2, and 2.

And in the same manner the proportional quantities of any number of ingredients may be found.

When the whole mixture is to be of a certain weight, find the quantity of each ingredient by the rule of proportion. Thus, suppose in the foregoing example a mixture of 30*lb.* is required.

Then,

$$\left.\begin{array}{l} \text{As } 7 + 2 + 2 \text{ or } 11 : \overset{lb.}{30} :: 7 : \overset{lb}{19\tfrac{1}{11}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\; 2 : 5\tfrac{5}{11} \\ \qquad\qquad\qquad\qquad\qquad\qquad\; 2 : 5\tfrac{5}{11} \end{array}\right\} \text{the } \textit{quantities required}$$

Questions of this kind when proposed to be solved arithmetically, come under the rule called *Alligation*. It is easy to perceive that they admit of a great variety of answers, which cannot however, be readily discovered without Algebra.

OF INTEREST.

103. INTEREST is the sum allowed for the loan or forbearance of money. It is reckoned at so much *per cent. per annum*, called the *rate*. Thus if £4 is paid for the use of £100 for 1 year, £4 is the interest, and the rate is 4 *per cent. per annum*. Or if £9 is paid for the use of £300 for a year, the rate of interest is 3 *per cent. per annum*; and

£
300 is the Principal or sum forborn.
9 is the Interest.
309 is the Amount.

Interest is distinguished into two kinds, *Simple*, and *Compound.*

106. *Simple Interest* is the allowance for the first sum or principal *only* for the whole time. So the simple interest of £100 for 3 years at 4 *per cent.* will be £12. Therefore the interest of any sum for a given time will be directly proportional to the principal.

Hence,

As £100
Is to its interest for any given time;
So is any other principal,
To its interest for that time.

Examples of Simple Interest.

1. What is the interest of £270 for 1 year at 4 *per cent*?

$$\text{As } \overset{£}{100} : \overset{£}{4} :: \overset{£}{270} : \frac{4 \times \overset{£}{270}}{100} = \overset{£}{10}\ \overset{s.}{16}. \quad \textit{Ans.}$$

2. What is the interest of £524 10s for 5 years at 3 *per cent*?

$$5 \times 3 = £15 \text{ the interest of } £100 \text{ for 5 years.}$$

$$\text{As } \overset{£}{100} : \overset{£}{15} :: \overset{£}{524\tfrac{1}{2}} : \overset{£}{78}\ \overset{s.}{13}\ \overset{d.}{6}. \quad \textit{Ans.}$$

3. How much is the interest of £122 15s. for 240 days at 5 *per cent*?

$$\text{As } \overset{d.}{365} : \overset{£}{5} :: \overset{d.}{240} : \overset{£}{\frac{5 \times 240}{365}} \text{ the interest of } £100 \text{ for 240 days.}$$

$$\text{As } £100 : \frac{5 \times 240}{365} :: £122\tfrac{3}{4} : \frac{5 \times 240 \times 122\tfrac{3}{4}}{100 \times 365} = \overset{£}{4.036}. \quad \textit{Ans.}$$

4. What will 218*l.* amount to in $2\frac{3}{4}$ years at $3\frac{1}{2}$ *per cent*?

$$2.75 \times 3.5 = 9.625 \text{ the interest of } £100 \text{ for 2.75 years.}$$
$$100$$
$$\text{Sum } \underline{109.625} \text{ amount of } £100 \text{ in 2.75 years.}$$

$$\text{As } \overset{£}{100} : \overset{£}{109.625} :: \overset{£}{218} : \overset{£}{238}\ \overset{s.}{19}\ \overset{d.}{7\tfrac{3}{4}}. \quad \textit{Ans.}$$

2. Required the discount of ... $2\frac{1}{2}$ years hence at 5 per cent?

... (£...) ... the interest or discount is required.

... the ... years

... the am... £100 ... years.

As ... the answer.

What is the ... stock at $106\frac{3}{4}$ per cent. or what ... must be ... stock.

As ... £2000 £127 10 *Ans*

... per cent. consols are done at ..., what is the interest of money?

As ... 100 ... per cent *Ans.*

COMPOUND INTEREST.

107. When the amount at Simple Interest is forborn, the interest arising from that sum is called Compound Interest. ... therefore ... succeeding amount may be found as in the ... example of Simple Interest, only repeating the operation.

Examples.

1. What is the amount of £120 in 4 years at 3 *per cent. per annum* compound interest?

The amount of £100 in 1 year is £103. Hence,

As £100 : £103 :: £120 : $\frac{103 \times 120}{100}$ the amount at the end of the 1st. year.

Or dividing the two first terms of the proportion by 100. (96).

As 1 : 1·03 :: 120 : 1·03 × 120, the amount at the end of the 1st. year.
1 : 1·03 :: 1·03 × 120 : 1·03 × 1·03 × 120 at the end of the 2d.
1 : 1·03 :: 1·03 × 1·03 × 120 : 1·03 × 1·03 × 1·03 × 120 at the end of the 3d.

As 1 : 1·03 :: 1·03 × 1·03 × 1·03 × 120 : 1·03 × 1·03 × 1·03 × 1·03, × 120, at the end of the 4th.

1·03 × 1·03 × 1·03 × 1·03 = 1·1255 (retaining 4 decimals only)
120

Ans. £135·0600, or £135 1·2s. the *amount.*

2. What is the compound interest of £242 10s. forborn $2\frac{1}{2}$ years at 4 *per cent. per ann.* the interest payable half yearly?

The interest of £100 for $\frac{1}{2}$ a year is £2.

Therefore the amount of £100 at the end of $\frac{1}{2}$ a year is £102

As £100 : £102 :: £242·5 :

Or dividing the two first terms by 100;

As 1 : 1·02 :: 242·5 : 1·02 × 242·5 the amount at the end of the first $\frac{1}{2}$ year.

And proceeding in the same manner for 5 half years, we have

1·02 × 1·02 × 1·02 × 1·02 × 1·02 × £242·5 for the whole amount.

1·02 × 1·02 × 1·02 × 1·02 × 1·02 = 1·10408 (retaining only 5 decimals)

And 1·10408 × 242·5 = £267·7394 the amount.
242·5 the principal, subtract,
25·2394 the *interest.*

But the operations in compound interest are much more expeditiously performed by means of Logarithms.

OF POSITION.

108. POSITION or the *Rule of False* is a method of solving questions by means of assumed or false numbers, and is of two kinds, *single*, and *double*.

Questions which require but one assumption, or where the results are proportional to the suppositions, belong to single position; such as the 20th. and 21st. examples in the Rules of Proportion.

DOUBLE POSITION.

109. WHEN two assumptions are made for answering the question, it is called Double Position; and sometimes the method of Trial-and-Error.

Rule. Make two suppositions, and proceed with each according to the conditions of the question. Then find the differences between the results and the result in the question.

Multiply the first supposition by the second difference or error; and the second supposition by the first difference or error.

Then, if the errors are *alike* (viz. both too great, or too little) divide the *difference* of the products by the *difference* of the errors, and the quotient will be the answer.

But if the errors are *unlike* (or one too great, and the other to little) divide the *sum* of the products by the *sum* of the errors for the answer.

This rule is founded on the supposition that the differences between the true and supposed numbers are directly proportional to the respective differences between the true and erroneous results (vol. 2. *art.* 128. *examp.* 8.) When that is not the case, the rule cannot give the exact answer.

Examples.

1. What two numbers are those whose sum is 10, and the greater divided by the less gives the quotient 20?

Suppose the numbers are	{ 20 1 }	the quotient being 20.
their sum	21	
But the sum should be	10	
Difference or *first error*	11	too great.

Again, suppose the numbers $\left\{ \begin{matrix} 40 \\ 2 \end{matrix} \right\}$ the quotient being 20.

their sum 42
But the sum should be 10
Difference or *second error* 32 *too great.*

Now the greater suppositions 20 and 40 will give the greater of the two required numbers; and the other suppositions 1 and 2 will find out the less.

First supposition 20
Second error . . 32
Product 640

Second supposition 40
First error 11
Product 440

640
440

Difference of products 200, which divided by 21 the *difference* of the errors 32 and 11, gives $\frac{200}{21} = 9\frac{11}{21}$ the greatest of the two required numbers.

Again (for the least of the two numbers).

First supposition 1
Second error . 32
Product 32

Second supposition 2
First error 11
Product 22

32
22

Difference of products 10 which divided by 21 the *difference* of the errors gives $\frac{10}{21}$ the least of the two required numbers. But when either number is found, the other will be given, because their sum is given.

To verify the rule when the errors are *unlike.* Let the first suppositions be 20 and 1 as before; then the *first error* will be 11 *too great.*

For the second suppositions $\left\{ \begin{matrix} 5 \\ 0\frac{1}{4} \end{matrix} \right\}$ the quotient being 20.
let the numbers be . ...
Sum . . $5\frac{1}{4}$
But the sum should be 10
Difference or *second error* $4\frac{3}{4}$ *too little.*

First supposition ... 20
Second error .... $4\frac{3}{4}$
Product 95

Second supposition 5
First error 11
Product 55

95
55
150 *sum* of products.

11
$4\frac{3}{4}$
$15\frac{3}{4}$ *sum* of errors.

Then $\frac{150}{15\frac{1}{2}} = 9\frac{21}{31}$ the greater number as before.

2. What number is that which added to its square shall make the sum 12?

Suppose the number to be	5	Next, suppose	4
Its square is 5×5 ...... =	25	Its square is $4 \times 4 =$	16
Sum .	30	Sum .. .	20
But the sum should be	12		12
Error	18 *too great.*	Error....	8 *too great.*

Products $\begin{cases} 5 \times 8 = 40 \\ 4 \times 18 = 72 \end{cases}$

Difference of errors 10) 32 *diff.* of products

3·2 quotient.

But the required number is 3, (for 3 added to 9, the square of 3, make 12) therefore the rule fails in this example. The true answer, however, may be approximated to any assigned degree of accuracy by repeating the operation, and constantly making the last quotients or approximations, the assumed numbers:

Thus,

Let one supposition be	4	And the other supp.	3·2
Its square (as before)	16	Its square $3{\cdot}2 \times 3{\cdot}2 =$	10·24
Sum	20	Sum	13·44
	12		12
Error as before	8 *too great.*	Error ...	1·44 *too great.*
		Error ...	8
		Diff. error ...	6·56

Products $\begin{cases} 4 \times 1{\cdot}44 = 5{\cdot}76 \\ 3{\cdot}2 \times 8 = 25{\cdot}6 \end{cases}$

19·84 *diff.* products.

$\frac{19{\cdot}84}{6{\cdot}56} = 3{\cdot}02$ the *second approximation.*

Next, making 3·2 and 3·02 the assumptions we have

$3{\cdot}02 \times 3{\cdot}02 = 9{\cdot}1204$ the square of 3·02

3·02

Sum 12·1404

12

Error 0·1404 *too great.*

1·44 first error, as before.

1·2996 *diff.* errors.

Products $\begin{cases} 3{\cdot}2 \times 0{\cdot}1404 = 0{\cdot}44928 \\ 3{\cdot}02 \times 1{\cdot}44 = 4{\cdot}3488 \end{cases}$

3·89952 *diff.*

Then $\frac{3·89952}{1·2996} = 3·0005$ the *third approximation.*

Again, let the suppositions be 3·02 and 3·0005; and the next approximation comes out 3·000001. And if the operation be repeated with 3·0005 and 3·000001, the result will be 3·00000000002, &c.

In this manner the rule may frequently be applied with success in very difficult cases.

OF INVOLUTION.

110. WHEN a number is multiplied into itself a certain number of times, it is called Involution, or raising of powers.

The number so multiplied is the root; and the products are the powers.

Thus if 2 be the root,

Then $2 \times 2 = 4$ is the 2*d* power or square of 2.
$2 \times 2 \times 2 = 8$ is the 3*d* power or cube of 2.
$2 \times 2 \times 2 \times 2 = 16$ is the 4*th* power or biquadrate.
$2 \times 2 \times 2 \times 2 \times 2 = 32$ is the 5*th* power or sursolid,
&c.

Roots	1	2	3	4	5	6	7	8	9
Squares	1	4	9	16	25	36	49	64	81
Cubes	1	8	27	64	125	216	343	512	729

111. The power to which a number is to be raised is usually denoted by a small figure called the *index* or *exponent.*

Thus 5^3 denotes the 3*d* power or cube of 5.
7^4 — the 4*th* power of [illegible]
10^2 — the square of 10. [illegible]
or *exponents* of the powers are 3, 4, and 2.

Since $2 \times 2 \times 2 \times 2 \times 2 = 32$ is the 5*th* [illegible] of the root 2, it follows that the 5*th* power is the product of [illegible] and cube.

For $2 \times 2 = 4$ is the square; and $2 \times 2 \times 2 = 8$ is the cube; therefore $4 \times 8 = 32$ the 5*th* power.

Hence $2^2 \times 2^3 = 2^5$; consequently the addition of the indices 2 and 3 answer to the multiplication of the powers; *viz.* $2^2 \times 2^3 = 2^{2+3}$.

Also $3^3 \times 3^4 = 3^7$. For 3^3 is 27; and 3^4 is 81; and 27×81 is equal to $2187 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^7$.

Other Examples.

What is the square of 100?

$$100 \times 100 = 10000.\ \textit{Ans.}$$

What is the square of $\frac{3}{11}$?

$$\frac{3}{11} \times \frac{3}{11} = \frac{9}{121}.\ \textit{Ans.}$$

Required the cube of the decimal ·013?

$$·013^3 = ·013 \times ·013 \times ·013 = ·000002197.\ \textit{Ans.}$$

What is the 4*th* power of 2·01?

$$2·01^2 = 2·01 \times 2·01 = 4·0401 \text{ the square, which}$$
squared is $2·01^4 = 4·0401 \times 4·0401 = 16·32240801$, the *Answer.*

EVOLUTION.

112. EVOLUTION is the extraction or finding the roots of any given powers, being the reverse of Involution.

Every number which is a known power will have a determinate or called a rational root: thus the number 8 is a cube number whose root is 2; and the number 9 is a square having 3 for the root: but 10 is not an exact power of any kind, because its root can never be accurately obtained. By the help of decimals however, the roots of any numbers may be approximated to any assigned degree of exactness: these approximate

roots are called *irrational* or *surd roots*. Thus any root of 10 will be a surd. And the square root of 8; and the cube root of 9 are both surds.

To Extract the SQUARE ROOT.

113. *Rule.* Begin at the units place and point the number into periods of two figures each.

Find the greatest square in the first period on the left hand and set its root on the right of the given number, in the same manner as a quotient figure in division.

Subtract the square from the period above it, and to the remainder bring down the next period, for a dividend.

Double the aforesaid root, and find how often it is contained in the dividend, exclusive of its right-hand figure, and set the result in the quotient, and also on the right of the divisor.

Multiply the augmented divisor by this last quotient figure, and subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.

Then find a new divisor by doubling the figures of the quotient; and proceed as before till all the periods are brought down.

The best way of doubling the root or quotient is by adding the last figure always to the last divisor.

Examples.

1. Required the square root of 41409225?

```
          41409225 ( 6435 root or quotient.
          36
   124 )  540
          [illegible]
   1283 ) 4492
      3   3849
   12865 ) 64325
           64325
```

114. The rule for extracting the square root is easily derived from the following method of forming a square or the product of two like numbers. For example, suppose 6435 × 6435 (the above square).

$$64 \times 64 \text{ is the sum of } \begin{cases} 60 \times 60 & \ldots\ldots\ldots = 3600 \\ 64 \times 4 \\ 60 \times 4 \end{cases} \Big\} = 124 \times 4 = 496$$

$$64 \times 64 = \underline{4096}$$

The sum of 64 × 4 and 60 × 4 being the same as 4 multiplied by twice 60 added to 4, or 124 × 4; therefore to find the difference of the squares of 60 and 64, add 4 to twice 60 and multiply the sum by 4.

In like manner the difference of the squares of 640 and 643 will be 3 added to twice 640 and the sum multiplied by 3, (1283 × 3).

And the difference between the squares of 6430 and 6435, is 5 added to twice 6430, and the sum multiplied by 5, or 12865 × 5; and so on.

Hence 6435 × 6435, or the square of 6435 will be

$$\text{the sum of } \begin{cases} 6000 \times 6000 = 36000000 \\ 12400 \times 400 = 4960000 \\ 12830 \times 30 = 384900 \\ 12865 \times 5 = 64325 \end{cases}$$

$$\underline{41409225}$$

Therefore as the whole square consists of the products 6000 × 6000, 12400 × 400, 12830 × 30, and 12865 × 5, if it be divided by 6000, and then the remainder by 12400, and the next remainder by 12830, and the last remainder by 12865, the quotients will be 6000, 400, 30, and 5, whose sum is the root.

```
 6000 ) 41409225 ( 6000
        36000000
12400 )  5409225 (  400
         4960000
12830 )   449225 (   30
          384900
12865 )    64325 (    5
           64325
                   ----
                   6435 the root.
```

In this operation the first divisor is the thousands in the root; the second is double the thousands added to the hundreds; the third is double the thousands and hundreds added to the tens; and the fourth is double the

thousands, hundreds, and tens, added to the units: hence the reason for doubling the root. And because a cipher in the divisor, and another in the quotient, will make two in the product, if the ciphers are omitted in both, it is evident that only two figures must be brought down at a time in order to form the dividend, which is the reason for pointing the number from the right to the left into periods of two figures each for it is manifest from the formation of the square, that the root will consist of as many figures as there are points or periods.

2. Required the square root of 100861849?

```
              100861849 ( 10043 root.
              1
     2004 )   008618
        4       8016
    20083 )      60249
                 60249
```

3. What is the square root of 5·9049?

```
         5·9049 ( 2·43 root.
         4
    44 ) 190
     4   176
   483 ) 1449
         1449
```

4. Required the square root of 8?

```
          8 ( 2·828 &c. root.
          4
    48  ) 400
     8    384
   562  ) 1600
     2    1124
  5648  ) 47600
     8    45184
           2416
```

Thus by annexing periods of two ciphers each to the remainders, the extraction may be continued to any number of decimals in the root. And the integral part of the root will consist of as many figures as there are points over the integers in the number whose root is required.

115. *The root of a proper fraction is greater than its square:* Therefore decimals are pointed at every second figure from the left-hand.

Required the square root of the decimal ·4.

```
       ·40 ( ·632 &c. root.
       36
 123 ) 400
   3   369
1262 ) 3100
       2524
        576
```

What is the square root of ·000950?

```
      ·000950 ( ·03082 &c. root.
          9
 608 )  5000
   8    4864
6162 )  13600
        12324
         1276
```

116. *To extract the square root of a Vulgar Fraction.* Reduce it to its lowest terms: then the roots of the numerator and denominator will form the fractional root required.

Thus the square root of $\frac{9}{16}$ is $\frac{3}{4}$.

And the square root of $\frac{50}{72}$ is $\frac{5}{6}$, for $\frac{50}{72} = \frac{25}{36}$ whose root is $\frac{5}{6}$.

Also, the square root of [illegible] is [illegible]; for [illegible] $=$ [illegible] whose root is [illegible].

When the terms of the fraction are not perfect squares, it may be reduced to a decimal, and its root extracted.

Thus, suppose the square root of $\frac{5}{7}$ is required?

$\frac{5}{7} = ·714285714$ &c. whose root is ·84515 &c.

Or because $\frac{5}{7} = \frac{5 \times 7}{7 \times 7} = \frac{35}{49}$, therefore the square root of 35 divided by 7 (the square root of 49) will be the root required.

The square root of 35 is 5·91608 nearly.

Therefore $\frac{5·91608}{7} = ·84515$ &c. the root, as before.

A *Mixt Number* may be brought to an improper fraction, and its root extracted as above.

Thus, to extract the square root of $11\frac{2}{3}$:

$$11\tfrac{2}{3} = \tfrac{35}{3}, \text{ which is equal to } \frac{35 \times 3}{3 \times 3} = \tfrac{105}{9} \text{ whose}$$

$$\text{root is } \frac{10{\cdot}2469 \text{ \&c}}{3} = 3{\cdot}41565 \text{ \&c. the root required}$$

Or the fraction may be reduced to a decimal, and the root of the whole extracted.

Thus $11\frac{2}{3} = 11{\cdot}6666$ &c whose *root* is $3{\cdot}41565$ &c.

To Extract the CUBE ROOT.

117. *Rule.* Point the number into periods of three figures each (beginning at the units) and find the greatest cube in the first period on the left hand, and set its root in the quotient for the first figure of the required root.

Subtract the cube from the period above it, and bring down the next period to the remainder for a dividend:

Divide this dividend by 300 times the square of the figure in the root, and the quotient figure will be the second figure in the root:

Subtract the cube of the two figures in the root from the two first periods on the left hand, and to the remainder bring down the next period for a new dividend:

Divide this dividend by 300 times the square of the two figures, and the quotient figure is the third figure in the root:

Subtract the cube of the three figures in the root from the three left hand periods; then proceed as before till all the periods are brought down.

N B. Should there be a remainder after all the figures of the proposed number are brought down, periods of 3 ciphers each may be annexed, and the root continued in decimals.

The [illegible] given in Art. 109, vol 2; only instead of neglecting two figures of the dividend on the right hand in making the division, the square of the root is multiplied by 300 (instead of 3) for the divisor.

Examples.

1. To extract the cube root of 4973940·243.

4973940·243 (170·7 *root.*
1
divisor $1^2 \times 300 = 300$) 3973 (7
4973 .... two first periods.
$17^3 =$ 4913
divisor $17^2 \times 300 = 86700$) 60940 (0
divisor $170^2 \times 300 = 8670000$) 60940243 (7
4973940243 . .. four periods.
$1707^3 =$ 4973940243
0

Here 1 is the greatest cube in 4 the first period.—3973 the first dividend, and 300 the first divisor; now 300 is contained more then 7 times in 3973, but $17^3 = 4913$ which is nearly equal to 4973 the two first periods and therefore a number or digit greater than 7 will not answer.

The 2*d.* dividend is 60940, and 86700 the 2*d.* divisor, consequently 0 is the 3*d.* figure in the root; in which case, another period is brought down to that dividend for a new dividend, and the three figures 170 are used in forming a new divisor.

In this example we have 3 points over the integers in the proposed number, and therefore the integral part of the root will consist of the like number of figures.

2. To extract the cube root of 6383500.

6383500 (185·5067 &c. *root.*
1
divisor $1^2 \times 300$) 5383 (8
6383 ... two first periods.
$18^3 =$ 5832
divisor $18^2 \times 300 = 97200$) 551800 (5
6383800 ....three first periods.
$185^3 =$ 6331625
divisor $185^2 \times 300 = 10267500$) 52175·000 (5
6383800·000 ... four periods.
$1855^3 =$ 6383101[illegible]
divisor $1855^2 \times 300 = 1032307500$) [illegible] (0
divisor $18550^2 \times 300 = 103230750000$) [illegible]
[illegible] periods.
63837[illegible]
divisor $185506^2 \times 300 = 10323742810800$) [illegible]465[illegible]

&c. &c.

The 4 decimals in the root are found by annexing 4 periods of three ciphers each.

3. Let the cube root of the decimal ·07 be required.

Here the periods or points are placed over every 3*d.* figure from the left hand.

$$
\begin{array}{rrl}
 & \cdot 070000 \text{ \&c.} & (\,412128 \text{ \&c. } \textit{root.} \\
 4^3 = & 64 & \\
 4^2 \times 300 = 4800\,) & 6000 & (\,1 \\
 & 70000 & \\
 41^3 = & 68921 & \\
 41^2 \times 300 = 504300\,) & 1079000 & (\,2 \\
 & 70000000 & \\
 412^3 = & 69934528 & \\
 412^2 \times 300 = 50923200\,) & 65472000 & (\,1 \\
 & 70000000000 & \\
 4121^3 = & 69985463561 & \\
 4121^2 \times 300 = 5094792300\,) & 14536439000 & (\,2 \\
 & 70000000000000 & \\
 41212^3 = & 69993653640128 & \\
 41212^2 \times 300 = 509528683200\,) & 6346359872000 & (\,3 \\
 \text{\&c.} & \text{\&c.} &
\end{array}
$$

The reason for pointing the number into periods of 3 figures each is manifest from the principles of common multiplication; for any number with one or more ciphers on the right hand, must have exactly 3 times as many ciphers in its cube.

118. But all the usual or common rules for extracting the cube and higher roots are extremely prolix. The following general method of approximation however, derived from the *rational formulæ* of Dr. Halley, (vol. 2, *art.* 111) is more expeditious, and easily remembered.

T[illegible] the Root of any Power.

Ass[illegible] the true root the better), then raise [illegible] whose root is required, and call it the [illegible]

Then take the sum of

The assumed power multiplied by its index added to 1;
And the given number multiplied by the index lessened by

And the sum of

The assumed power multiplied by the index lessened by 1;
And the given number multiplied by the index added to 1

Then say, by the Rule of Proportion,
As the first of those sums,
Is to the second,
So is the assumed root,
To the required root, nearly. And if this root be taken for the assumed root, and the operation repeated, a nearer approximation will be obtained; and so on.

Examples of the Cube Root.

1. Required the 3*d*. or cube root of 184?

Assume 6 for the root, whose cube is 216, the assumed power. Then the index 3 added to 1, and lessened by 1, give 4 and 2.

Therefore,

As the sum of 216×4 and 184×2,
Is to the sum of 216×2 and 184×4;
So is the assumed root 6,
To the root, nearly.

Or dividing the two first terms of the proportion by 2 we have (96.)

As the sum of 216×2 and 184,
Is to the sum of 216 and 184×2;
So is 6,
To the root, nearly.

In words,

As twice the assumed cube added [illegible],
Is to the assumed cube added [illegible]
So is the assumed root,
To the required root, nearly.

Assumed cube............	216	Given number..........	184
	2		2
	432		368
Given number...	184	Assumed cube	216
Sum .	616	*Sum*	584

As 616 : 584 :: 6 : 5·7 *root nearly*.

Now taking 5·7 for the assumed root, its cube is 185·193 the assumed cube.

Assumed cube............	185·193	Given number...... ..	184
	2		2
	370·386		368
Given number.............	184	Assumed cube...	185·193
Sum........	554·386	*Sum*	553·193

As 554·386 : 553·193 :: 5·7 : 5·687734 *root*, which is true in the last decimal.

2. Required the cube root of the decimal ·07 ?

Assume ·4 for the root, its cube being ·064.

	·064		·07
	2		2
	·128		·14
	·07		·064
Sum	·198	*Sum*	·204

As ·198 : ·204 :: ·4 : ·41 *root* nearly.

Now take ·068921 the cube of ·41 for the second assumed cube.

	·068921		·07
	2		2
	·137842		·14
	·07		·068921
Sum .	·207842	*Sum*...	·208921

As ·207842 : ·208921 :: ·41 : ·4121285 *root*, true to the last figure.

For $·4121285^3$ = ·06999998 + (retaining 8 places of decimals only) which is less than ·00000002 short of the truth.

110. *To extract the cube root of a Vulgar Fraction.* Reduce it to its lowest terms; then the roots of the numerator and denominator will form the fractional root required.

[illegible] root of $\frac{1}{1000}$ is $\frac{1}{10}$.

[illegible] cube root of $\frac{[illegible]}{[illegible]}$ is $\frac{[illegible]}{[illegible]}$, for [illegible] cube root is $\frac{[illegible]}{[illegible]}$.

But when the terms of the fraction are not perfect cubes, let them be multiplied by the square of the denominator, then extract the root of the new fraction for the root required.

Thus, suppose the cube root of $\frac{3}{7}$ is required.

The fraction $\frac{3}{7}$ is $= \frac{3 \times 49}{7 \times 49} = \frac{147}{343}$ whose cube root is $\frac{5\cdot27763 \text{ \&c.}}{7} =$ ·75394 &c. the root required.

Or the fraction may be reduced to a decimal, [illegible] numbers are prepared as in extracting the square root.

120. The *Biquadratic*, or *4th root*, is obtained by extracting the square root, and then extracting the square root of that root.

Thus the *4th* root of 6561 is 9. For the square root of 6561 is 81, whose square root is 9.

Let the *5th* root of 27 be required.

Assume 2 for the root; then its *5th* power is 32.

And the index 5 added to 1, and lessened by 1 give 6 and 4.

Then $32 \times 6 = 192$	$32 \times 4 = 128$
$27 \times 4 = \underline{108}$	$27 \times 6 = \underline{162}$
Sum 300	Sum 290

As 300 : 290 :: 2 : 1·93, *root nearly*, the first approximation.

Now assume 1·93 for the root; then its *5th* power, or the assumed power is 26·778 (retaining 3 places of decimals only).

$26\cdot778 \times 6 = 160\cdot668$	$26\cdot778 \times 4 = 107\cdot112$
$27 \times 4 = \underline{108}$	$27 \times 6 = \underline{162}$
Sum $268\cdot668$	Sum $269\cdot112$

As 268·668 : 269·112 :: 1·93 : [illegible]

OF ARITHMETICAL PROPORTION AND PROGRESSION.

121. WHEN four numbers have a common difference they are said to be in continued arithmetical proportion. But if the difference of the first and second is equal to the difference of the third and fourth, but not to that between the second and third, it is called discontinued proportion.

> 2, 4, 6, 8, continued proportion.
> 2, 4, 7, 9, discontinued proportion.

122. A series or rank of the first kind form a progression:

$$\left.\begin{matrix} 1, 2, 3, 4, 5, 6, \&c. \\ 0, \tfrac{1}{2}, 1, 1\tfrac{1}{2}, 2, 2\tfrac{1}{2}, \&c. \end{matrix}\right\} \text{ascending series or progressions.}$$

$$\left.\begin{matrix} 32, 29, 26, 23, 20, 17, \&c. \\ 10\tfrac{1}{2}, 10\tfrac{1}{4}, 10, 9\tfrac{3}{4}, 9\tfrac{1}{2}, 9\tfrac{1}{4}, \&c. \end{matrix}\right\} \text{descending progressions.}$$

123. The first and last numbers or terms are called the extremes; and the others between them the means.

Thus 1 and 6 are the extremes; and, 2, 3, 4, 5, the means of the rank 1, 2, 3, 4, 5, 6.

124. It is evident from the nature of the progressions, that the double of any term is equal to the sum of the two adjacent terms, or to the sum of any two terms equidistant from it.

> Thus in the rank 1, 2, 3, 4, 5, 6, &c.
> twice $4 = 3 + 5 = 2 + 6$.

125. Hence if three numbers are in arithmetical proportion, twice the mean is equal to the sum of the two extremes.

> Thus, if the three numbers are $10\tfrac{1}{2}$, 10, $9\tfrac{1}{2}$,
> Then $10 \times 2 = 10\tfrac{1}{2} + 9\tfrac{1}{2}$.

126. And when 4 numbers are in arithmetical proportion, the sum of the two means is equal to that of the extremes.

Thus if 32, 29, 20, 17, are the 4 numbers,
Then $29 + 20 = 32 + 17$.

127. Since the terms of an arithmetical progression are found by continually adding or subtracting the common difference; if the difference, twice the difference, three times the difference, &c. be added to the first term, the several sums will give an ascending series; or subtracted, a descending one.

Thus the terms of the progression 3, 5, 7, 9, 11, &c. having the common difference 2, will be

$$3,\ 3+2,\ 3+4,\ 3+6,\ 3+8,\ \&c.$$

And the terms of the series 6, $5\frac{1}{2}$, 5, $4\frac{1}{2}$, 4, &c. where the common difference is $\frac{1}{2}$,

$$\text{is } 6,\ 6-\tfrac{1}{2},\ 6-1,\ 6-1\tfrac{1}{2},\ 6-2,\ \&c.$$

128. Consequently when the first and last terms are given, if their difference be divided by the number of terms lessened by 1, the quotient will be the common difference of the terms.

For example, let the first term be 2, the last 20, and the number of terms 7.

Then $20 - 2 = 18$ the difference, which divided by 6 (or $7 - 1$) gives 3, for the common difference of the terms. And the progression will be

$$2,\ 5,\ 8,\ 11,\ 14,\ 17,\ 20.$$

129. In this manner we can find any proposed number of arithmetical means between two given numbers; or interpose any number of terms between two given extremes.

For example, let 9 arithmetical means be found between 1 and 2.

Now the whole number of terms being 11, that number lessened by 1 is 10:

And $2 - 1 = 1$ the difference of the extremes, which divided by 10 gives $\frac{1}{10}$ the common difference of the terms.

And the series will be

$$1,\ 1\tfrac{1}{10},\ 1\tfrac{2}{10},\ 1\tfrac{3}{10},\ 1\tfrac{4}{10},\ 1\tfrac{5}{10},\ 1\tfrac{6}{10},\ 1\tfrac{7}{10},\ 1\tfrac{8}{10},\ 1\tfrac{9}{10},\ 2.$$

130. Hence it appears that the difference of the extremes, divided by the common difference of the terms, gives the number of terms less by 1.

For example, let the extremes be 2 and 20, and 3 the common difference.

Then $\frac{20-2}{3} = 6$; therefore $6 + 1 = 7$ the number of terms.

131. Therefore it is evident that the number of terms less by 1, multiplied by the common difference, is equal to the difference of the two extremes.

Thus if the number of terms be 7, and the common difference 3;
Then $7 - 1 = 6$ the number of terms less by 1;
And $6 \times 3 = 18$ the difference of the extremes; which added to the less extreme will give the greater; or subtracted from the greater will give the less.

132. The sum of all the terms, in a continued arithmetical series or progression, is equal to the sum of the two extremes, multiplied into half the number of terms.

Examples.

1. Required the sum of 2, 4, 6, 8, 10, 12?

[illegible] 12 to these add the same series in an inverted order
[illegible] 14 14. Now the sum of these numbers is evidently equal to twice the proposed series:

But their sum is 14×6 (or 84) or the sum of the first and last terms multiplied by the number of terms.

Therefore half that sum or the sum of the series is $14 \times 3 = 42$, viz. the sum of the two extremes into half the number of terms.

[illegible] 000 stones be placed on the ground in a direct line at the [illegible] yard from each other; how far would a person travel in [illegible] one at a time, to a basket placed a yard behind the [illegible]

The distance for the first stone will be 2 yards, and that for the last 2000; which therefore, are the two extremes

2002	sum of extremes.
500	half the number of terms.
1001000	yards, or 568¾ miles, the *answer*.

Of GEOMETRICAL PROPORTION AND PROGRESSION.

133. In arithmetical proportion numbers are compared by means of their differences; but in geometrical proportion by the quotient arising from the division of one number by another. Thus, when the quotients are equal, the numbers which produce them are said to be in geometrical proportion. For example, the numbers 2, 4, 5, 10, are in geometrical proportion, because $\frac{2}{4} = \frac{5}{10}$; see *art.* 92, &c. What we have to add concerning proportion chiefly relates to the permutation, composition, &c. of the terms, and ratios.

134. In any number of proportionals taken two and two in order, the first, third, fifth, &c. terms are called antecedents; and the second, fourth, sixth, &c. their consequents.

Thus, if the terms are 2 : 4 :: 5 : 10 :: 9 : 18, then 2, 5, 9 are the antecedents; and 4, 10, 18 their consequents.

135. When 4 numbers are proportional, [illegible] of 8 variations or permutations.

Let the numbers be 3, 5, 9, 15.

Then $\frac{3}{5} = \frac{9}{15}$
$\frac{5}{3} = \frac{15}{9}$
$\frac{3}{9} = \frac{5}{15}$
$\frac{9}{3} = \frac{15}{5}$

Therefore (92.) 3 : 9 :: 5 : 15
9 : 3 :: 15 : 5
5 : 15 :: 3 : 9
15 : 5 :: 9 : 3
3 : 5 :: 9 : 15
5 : 3 :: 15 : 9
9 : 15 :: 3 : 5
15 : 9 :: 5 : 3

136. In a rank of proportionals standing in order, two and two.—As any antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let the proportionals be $3 : 5 :: 9 : 15 :: 36 : 60$.

Then $3 : 5$ (or $9 : 15$) $:: 3 + 9 + 36 : 5 + 15 + 60$

or $3 : 5 :: 48 : 80$.

For $3 : 3 :: 5 : 5$, hence $\frac{3}{3} = \frac{5}{5}$.

$3 : 9 :: 5 : 15$, hence $\frac{9}{3} = \frac{15}{5}$.

$3 : 36 :: 5 : 60$, hence $\frac{36}{3} = \frac{60}{5}$.

&c. &c.

Now the sums of the equal fractions must also be equal,

$$viz.\ \frac{3 + 9 + 36}{3} = \frac{5 + 15 + 60}{5};$$

Therefore (92) $3 : 5 :: 3 + 9 + 36 : 5 + 15 + 60$.

This is called *composition* of proportion.

137. If 4 numbers are proportional, then, as the difference of the first and second, is to the first (or second), so is the difference of the third and fourth, to the third (or fourth).

Suppose $3 : 5 :: 9 : 15$

Then $5 - 3 : 3 :: 15 - 9 : 9$

And $5 - 3 : 5 :: 15 - 9 : 15$.

For $\frac{5}{3} = \frac{15}{9}$; and if we take $\frac{3}{3}$ (or 1) from $\frac{5}{3}$ the remainder is $\frac{5 - 3}{3}$.

And $\frac{9}{9}$ (or 1) taken from $\frac{15}{9}$ leaves $\frac{15 - 9}{9}$.

And since equal numbers subtracted from equal numbers must give equal remainders, the fractions $\frac{5 - 3}{3}$, $\frac{15 - 9}{9}$ must be equal.

Therefore (92) $5 - 3 : 3 :: 15 - 9 : 9$.

This is called *division* of proportion.

138. Since $3 : 5 :: 9 : 15$, and (by composition) $5 + 3 : 3 :: 15 + 9 : 9$; therefore $5 + 3$ and $15 + 9$ have the same proportion as $5 - 3$ and $15 - 9$ (137). Hence when 4 numbers are proportional; As the sum of the first and second is to their difference, so is the sum of the third and fourth, to their difference.

$5 + 3 : 5 - 3 :: 15 + 9 : 15 - 9$

or $8 : 2 :: 24 : 6$

139. If several numbers are proportionals, their squares, cubes, &c. are proportionals.

For example, suppose $3 : 5 :: 9 : 15$

Then $\frac{3}{5} = \frac{9}{15}$; now those fractions being equal, their like powers must be equal,

$$viz. \ \frac{3^2}{5^2} = \frac{9^2}{15^2}$$

$$\text{and } \frac{3^3}{5^3} = \frac{9^3}{15^3}, \text{ \&c.}$$

Therefore (92) $3^2 : 5^2 :: 9^2 : 15^2$
or $9 : 25 :: 81 : 225$

And $3^3 : 5^3 :: 9^3 : 15^3$
or $27 : 125 :: 729 : 3375$, &c.

Hence the square, cube, &c. roots of proportional numbers, are also proportional.

140. If there are several ranks of proportionals standing in order two and two, the products of the corresponding terms will be proportional.

For example, let $\left.\begin{array}{l} 3 : 5 :: 9 : 15 \\ 12 : 6 :: 8 : 4 \end{array}\right\}$ be two ranks

then $3 \times 12 : 5 \times 6 :: 9 \times 8 : 15 \times 4$
or $36 : 30 :: 72 : 60$.

For $\frac{3}{5} = \frac{9}{15}$; and $\frac{12}{6} = \frac{8}{4}$. And since equal numbers multiplied by equal numbers must give equal products, $\frac{3}{5} \times \frac{12}{6}$ must be equal to $\frac{9}{15} \times \frac{8}{4}$, or $\frac{3 \times 12}{5 \times 6} = \frac{9 \times 8}{15 \times 4}$; therefore (92) $3 \times 12 : 5 \times 6 :: 9 \times 8 : 15 \times 4$; and so of any other number of ranks.

141. Hence the ratio of the products is compounded of the ratios of the terms;

For $\frac{3}{5}$ denotes the ratio of 3 to 5; and $\frac{12}{6}$ that of 12 to 6;

And the product $\frac{3 \times 12}{5 \times 6}$ denotes the ratio of $3 \times$ [illegible] of the other terms.

Therefore ratios are compounded by multiplying together the fractions denoting those ratios.

PROGRESSION.

142. The terms of a geometrical progression result from successive multiplications, or divisions, by some number, which is called the common ratio of the terms.

Thus, if 1 be the first term, and 2 the ratio;
Then 1, 2, 4, 8, 16, 32, &c. is an ascending progression.
And 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c. a descending progression.

143. The first and last terms are called the extremes; and the intermediate ones the geometrical means.

144. In any continued geometrical series, the product of the two extremes is equal to that of any two means equally distant from them.

Thus, if the series be 2, 4, 8, 16, 32, 64;
Then $2 \times 64 = 4 \times 32 = 8 \times 16$.

For the ratio of every two adjacent terms being the same, we have $2 : 4 :: 32 : 64$

Therefore $2 \times 64 = 4 \times 32$.

The terms $2 : 4 :: 32 : 64$ are said to be in discontinued proportion, because the ratio of the first and second terms (2, 4,) and that of the second and third (4, 32,) are unequal.

145. In any [illegible] geometrical series, the ratio of the first term [illegible] compounded of the ratios of all the antecedents [illegible]

[illegible] progression 1, 2, 4, 8, 16, 32, the [illegible] ratios [illegible]

[illegible]1) the compounded ratio is $\frac{1 \times 2 \times 4 \times 8 \times 16}{2 \times 4 \times 8 \times 16 \times 32}$, which fraction in its lowest terms is $\frac{1}{32}$, denoting the ratio of 1 to 32.

146. All the terms of a geometrical progression may be expressed by means of the common ratio and one of the extremes.

Thus, the series 3, 6, 12, 24, 48, &c. where the common ratio is 2, and first term 3, will be

$3,\ 3 \times 2,\ 3 \times 2 \times 2,\ 3 \times 2 \times 2 \times 2,\ 3 \times 2 \times 2 \times 2 \times 2$, &c.
or $3,\ 3 \times 2,\ 3 \times 2^2,\ 3 \times 2^3,\ 3 \times 2^4$, &c. (111)

147. Therefore in any ascending progression, if the first term be multiplied by the ratio raised to the power whose index is the number of terms less by 1, the product will be the last term.

For example, suppose the first term is $\frac{1}{4}$, the common ratio 2, and the number of terms 10; what is the last term?

The number of terms less by 1 is 9

And $2^9 = 512$, which multiplied by $\frac{1}{4}$ (the first term) gives 128 the last term.

148. But in a descending progression (where the terms result from division) the first term divided by the said power of the ratio gives the last term.

Thus, suppose the first term is 128, the common ratio 2, and the number of terms 10, what is the last term?

$2^9 = 512$, and 128 divided by 512 gives $\frac{128}{512}$ or (in its lowest terms) $\frac{1}{4}$ the last term.

149. Hence, if one extreme be divided by the other, the quotient will be that power of the ratio whose index is the number of terms less by 1; and consequently its root will be the ratio.

For example, if 7 be the first term, 189 the last, and 4 the number of terms; what is the ratio?

$\frac{189}{7} = 27$ the 3d. power of the ratio (the number of terms [illegible]), whose cube root is 3 the ratio required.

Therefore the 4 terms are 7, 7 × 3, 7 × 3², 189.
or 7, 21, 63, 189.

150. In like manner we find a proposed number of geometrical mean proportionals between two given numbers.

For example, let it be required to find 3 geometrical means between 6 and 1536.

$\frac{1536}{6} = 256$ the *4th.* power of the ratio (the number of terms being 5).

The square root of 256 is 16 whose square root is 4, the *4th* root of 256 or the required ratio.

And the three means will be 6 × 4, 6 × 4², 6 × 4³;
or 24, 96, 384;

And the series 6, 24, 96, 384, 1536.

151. When only one mean proportional between two given numbers is required, the square root of their product will be the answer.

For example, to find a mean proportional between 8 and 18.

8 × 18 = 144 whose square root is 12 the *answer*.

For 8 : 12 :: 12 . 18.

And 18 is called a third proportional to 8 and 12.

152. To find the sum of all the terms in a given progression; suppose 2, 6, 18, 54, 162; where the common ratio is 3.

2, 6, 18, 54, 162
3
6, 18, 54, 162, 486 the series multiplied by the ratio 3.
6, 18, 54, 162. .... the series itself, subtract:
. . . . 486 lessened by 2 is the remainder.

T[illegible]der is equal to twice the sum of the series, because it is the diffe[illegible]ween the series and three times the series.

Therefore if 486 less by 2, be divided by 2 (*viz.* the ratio less by 1) the quotient will be the sum of the series.

But 486 less by 2 is the difference between the first term, and the product of the last by the ratio: hence the following

Rule. Multiply the last term by the ratio, and take the first term from the product, then divide the difference by the ratio lessened by 1, and the quotient is the sum of the progression.

In a descending progression take the first term for the last, and *vice versâ.*

Ex 2. Required the sum of the series 65536, 16384, 4096, &c. continued to 12 terms?

The ratio or divisor is 4; and $4^{11} = 4194304$.

And 65536 divided by 4194304 gives $\frac{65536}{4194304}$ or (in its lowest terms) $\frac{1}{64}$ the *12th.* or last term of the series, which being made the first term, and 65536 the last, the work will stand as below.

$$
\begin{array}{rl}
65536 & \\
4 & \text{ratio.} \\
\hline
262144 & \\
\frac{1}{64} & \text{subtract.} \\
\hline
\text{the ratio less by } 1 = 3\,)\ 262143\frac{[illegible]}{64} & \\
\hline
87381\frac{[illegible]}{64} & \textit{sum of the series.}
\end{array}
$$

3. An officer with a detachment of 60 men having taken a very strong fort by surprize, desired as a reward for himself and the party, 1 musket bullet for the first man, 2 for the second, 4 for the third, 8 for the fourth, and so on, doubling to 60 times (the number of men) Now suppose each bullet to be an ounce, and the lead at 5 shillings the hundred weight; what would be the value of his request?

Here the first term is 1, the ratio 2, and the number of terms 60; therefore 2^{59}, or 2 raised to the *59th.* power will be the last term of the series.

The *6th.* power of 2 is 64, which cubed is 262144 the [illegible], power (111,) and that cubed gives 18014398509181984 the *54th.* [illegible], which

multiplied by 32 (the *5th.* power of 2) is 576460752303423488 *the 59th.* power or last term of the series; this multiplied by 2 the ratio, and 1 (the first term) subtracted from the product, gives 1152921504606846975 *the sum of the series*, or *number of bullets*, or *ounces* (because the ratio lessened by 1 is 1), equal to 643371375338642$\frac{511}{1792}$ hundred weight, which at 5 shillings the hundred, amounts to £160842843834660$\frac{595}{1792}$ the *answer*.

ADDITIONAL EXAMPLES

IN THE

Foregoing RULES of ARITHMETIC.

Vulgar Fractions.

153. Required the greatest common measure

of 172[illegible], 1458 .... *Ans.* 54.
1400, 35000 700
1353, 1419, 187 .... 11.
2678, 4056, 6708, 7917 .... 13.
9057, [illegible].
219, 9101.
10307, 842[illegible], 937.
5600, 670[illegible], 103[illegible].

Reduce to the lowest terms

[illegible] .... *Ans.* $\frac{1}{6}$, $\frac{2}{3}$, $\frac{11}{13}$.
[illegible] .... $\frac{17}{19}$, $\frac{21}{16}$.
[illegible].

Reduce to equivalent whole or mixt numbers

[illegible] ..... *Ans.* $1\frac{4}{5}$, [illegible], 8.
[illegible] ..... [illegible], [illegible], 2940[illegible]

Reduce to improper fractions

$11\frac{1}{5}$, [illegible] ..... *Ans.* [illegible], $\frac{127}{10}$, $\frac{22}{21}$.
[illegible] ..... [illegible], $\frac{10001}{10}$, $\frac{10000}{999}$.

Reduce to simple fractions

[illegible] of [illegible] of $\frac{6}{7}$ .... *Ans.* $\frac{3}{7}$.
[illegible] of $\frac{2}{7}$ of [illegible] .... [illegible].
$\frac{3}{7}$ of [illegible] of [illegible] of 7 .... [illegible].
[illegible] of [illegible] of [illegible] .... [illegible].
[illegible] of $\frac{3}{4}$ of 10 of [illegible] .... [illegible].
$\frac{27}{56}$ of [illegible] of [illegible].
$\frac{5}{6}$ of $1\frac{1}{2}$ of $\frac{4}{5}$ of [illegible] of $1\frac{2}{5}$.

Required the least common multiple of the nine digits, or the least whole number that is divisible by 1, 2, 3, 4, 5, 6, 7, 8, and 9, without leaving a remainder? *Ans.* 2520.

Of 10, 35, 15, and 12 .... *Ans.* 1260.
Of 50, 120, 76, and 59.
Of 162, 27, 729, 486.

Required the least common multiple of $10\frac{1}{2}$, $13\frac{3}{4}$, and $26\frac{1}{4}$? *Ans.* $2782\frac{1}{2}$

Reduce to the least common denominators

$\frac{3}{7}$, $\frac{5}{9}$, $\frac{2}{5}$ .... *Ans.* $\frac{135}{315}$, $\frac{175}{315}$, $\frac{126}{315}$.
$\frac{2}{5}$, $\frac{7}{10}$, $\frac{13}{20}$ .... $\frac{8}{20}$, $\frac{14}{20}$, $\frac{13}{20}$.
$\frac{3}{8}$, $\frac{3}{4}$, $\frac{7}{10}$, $\frac{7}{12}$ .... $\frac{45}{120}$, $\frac{90}{120}$, $\frac{84}{120}$, $\frac{70}{120}$.
$2\frac{1}{2}$, $\frac{7}{9}$, 6 .... $\frac{45}{18}$, $\frac{14}{18}$, $\frac{108}{18}$.
$\frac{9}{10}$, *and* $\frac{2}{5}$ *of* $5\frac{2}{5}$ .... $\frac{45}{50}$, $\frac{108}{50}$.
$\frac{4}{5}$ *and* 12 .... $\frac{4}{5}$, $\frac{60}{5}$.
$\frac{91}{111}$ *and* $\frac{37}{40}$.
$\frac{81}{729}$, $\frac{729}{2187}$, $\frac{162}{486}$.
$\frac{84}{312}$, $\frac{35}{252}$, $\frac{475}{256}$.

Addition.

Required the sums of

$\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$ .... *Ans.* $1\frac{1}{2}$.
$\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{3}$ .... [illegible]
$\frac{7}{9}$, $\frac{2}{9}$, $\frac{1}{9}$ .... $1\frac{1}{9}$
$\frac{2}{11}$, $\frac{2}{11}$, $\frac{7}{11}$ .... 1.
$\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$ .... 2.
$\frac{1}{2}$, $\frac{1}{3}$ .... $\frac{5}{6}$.
$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ .... $1\frac{17}{60}$.
$\frac{2}{3}$, $\frac{11}{15}$, $\frac{3}{5}$ .... 2.
$\frac{5}{9}$ *and* $\frac{2}{3}$ *of* $18\frac{1}{2}$ .... $12\frac{8}{9}$.
$\frac{3}{4}$ *of* 10, $\frac{1}{2}$ *of* 1[illegible], *and* $\frac{1}{4}$ *of* 11 .... 17[illegible].
$74\frac{4}{5}$, $274\frac{4}{5}$ .... $319\frac{3}{5}$.
$96400\frac{5}{6}$, $11\frac{7}{9}$ .... $96412\frac{11}{18}$.
$7162\frac{2}{3}$, *and* $\frac{1}{3}$ *of* $5846\frac{1}{2}$ .... $9411\frac{1}{6}$.
$100\frac{2}{3}$, $2000\frac{3}{4}$, $1764\frac{4}{5}$ .... $3860\frac{13}{60}$.
$1000\frac{152}{1312}$, [illegible]$99\frac{298}{3076}$.
[illegible] $\frac{7}{5}$, $\frac{9}{10}$, $\frac{11}{12}$.
[illegible]$9\frac{1}{2}$, $111\frac{19}{192}$, $1013\frac{24}{129}$, $17\frac{2}{3}$.
$10\frac{1}{20}$, $100\frac{1}{30}$, $301\frac{1}{16}$, $5111\frac{1}{12}$.
[illegible], [illegible], [illegible], $20\frac{5}{7}$[illegible].

Subtraction.

Required the differences

of $\frac{2}{7}$, $\frac{1}{7}$ .... *Ans.* $\frac{1}{7}$.
$\frac{3}{8}$, $\frac{5}{8}$ .... $\frac{1}{4}$.
$\frac{13}{14}$, $\frac{9}{14}$ .... $\frac{2}{7}$.
19, $9\frac{5}{11}$ .... $9\frac{6}{11}$.
$19\frac{6}{11}$, $9\frac{5}{11}$ .... $10\frac{1}{11}$.
$19\frac{6}{11}$, $9\frac{8}{11}$ .... $9\frac{9}{11}$.
$\frac{1}{2}$, $\frac{1}{3}$ .... $\frac{1}{6}$.
$\frac{1}{2}$, $\frac{2}{3}$ .... $\frac{1}{6}$.
$\frac{4}{5}$, $\frac{3}{4}$ .... $\frac{1}{20}$.
$\frac{5}{6}$, $\frac{1}{3}$ .... $\frac{1}{2}$.
$\frac{791}{1245}$, $\frac{7}{11}$.
$1000\frac{3}{5}$, $100\frac{14}{15}$ .... $899\frac{2}{3}$.
$10\frac{1}{2}$, *and* $\frac{5}{6}$ *of* $10\frac{1}{2}$ .... $1\frac{3}{4}$.
$\frac{2}{3}$ *of* 8, *and* $\frac{2}{3}$ *of* 7 .... $\frac{2}{3}$.
$\frac{3}{5}$ *of* $\frac{10}{11}$, *and* $\frac{2}{3}$ *of* $\frac{10}{11}$ .... $\frac{2}{33}$.
10, *and* $\frac{10}{11}$ *of* 10 .... $\frac{10}{11}$.
10000 *and* $999\frac{1}{999}$.
$\frac{5001}{5005}$ *and* 1.

Multiplication.

Required the products

of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$ .... *Ans.* $\frac{1}{4}$.
$\frac{5}{9}$, $\frac{9}{20}$ .... $\frac{1}{4}$.
$2\frac{2}{3}$, $\frac{3}{8}$ .... 1.
$\frac{2}{3}$ *of* $\frac{3}{4}$, *and* $\frac{4}{5}$ *of* $\frac{5}{6}$ .... $\frac{1}{3}$.
$3\frac{3}{4}$, $3\frac{1}{3}$, $1\frac{1}{2}$ .... $18\frac{3}{4}$.
20, $10\frac{1}{2}$, $\frac{3}{5}$, $\frac{5}{9}$ .... 70.
$\frac{1}{16}$, 8 .... $\frac{1}{2}$.
22, $\frac{3}{11}$ .... 6.
22 $\frac{3}{10}$ .... $6\frac{3}{5}$.
$2\frac{1}{2}$, $4\frac{1}{2}$, $\frac{5}{9}$, $\frac{4}{5}$, 10, *and* $\frac{2}{3}$ *of* $\frac{3}{4}$ .... 25.
$\frac{1}{3}$ *of* 20, *and* $\frac{3}{4}$ *of* 30 .... 150.
$\frac{2}{3}$ *and* $74131\frac{1}{2}$ .... 49421.
$\frac{1}{2}$, $\frac{1}{3}$, 56427 .... $9404\frac{1}{2}$.
$646124\frac{1}{2}$, $64\frac{1}{2}$ .... $41675030\frac{1}{4}$.
84672, $1000\frac{1}{9}$ .... 84681408.
$8320\frac{1}{111}$, $\frac{37}{29731}$ .... $10\frac{1}{3}$.

$64120\frac{2}{3}$, $101\frac{1}{4}$......................*Ans.* $6492217\frac{1}{2}$.
$\frac{2}{3}$, $\frac{9}{11}$, $\frac{7}{27}$, $\frac{22}{63}$, *and* $1\frac{1}{6}$.
$\frac{17}{252}$, $5\frac{16}{17}$, and 1000.
$5734\frac{3}{7}$, 100, 1492[illegible].
$1\frac{1}{11}$, $1\frac{1}{9}$, $\frac{9}{10}$, *and* $\frac{11}{12}$.
$\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, *and* 5040.

Division.

Divisors.	*Dividends.*	*Quotients.*
$\frac{7}{9}$,	$\frac{35}{72}$	$\frac{5}{8}$.
$\frac{2}{3}$,	$\frac{64}{81}$	$1\frac{5}{27}$.
$\frac{1}{6}$,	$\frac{1}{18}$	$\frac{1}{3}$.
$\frac{29}{39}$,	1	$\frac{39}{29}$.
$\frac{21}{27}$,	$\frac{7}{9}$	1.
$\frac{3}{5}$,	$\frac{6}{11}$	$\frac{10}{11}$.
$2\frac{2}{3}$,	1	$\frac{3}{8}$.
$\frac{9}{20}$,	$\frac{1}{4}$	$\frac{5}{9}$.
$3\frac{1}{2}$,	$12\frac{1}{4}$	$3\frac{1}{2}$.
$\frac{5}{7}$,	10	14.
10,	$\frac{5}{7}$	$\frac{1}{14}$.
$37065\frac{1}{2}$,	[illegible]	$\frac{2}{3}$.
$\frac{1}{6}$,	[illegible]10 $\frac{1}{2}$	[illegible]7.
$421\frac{3}{15}$	2101 [illegible]	[illegible]7.
$\frac{3}{4}$ of 30	[illegible]	[illegible]
$\frac{2}{3}$ of $\frac{3}{4}$,	[illegible] of 6	$1\frac{1}{3}$.
$\frac{5}{7}$ of 9,	$\frac{4}{7}$ of 9	$\frac{4}{5}$.
$\frac{2}{3}$,	$10000\frac{2}{3}$	15001.
7,	416121[illegible] $\frac{7}{48}$	[illegible] $\frac{5}{336}$.

Divide the difference of $3\frac{2}{5}$ and [illegible] by the sum....1[illegible].

1263[illegible] *by* 210[illegible].
$\frac{1}{16384}$ *by* $\frac{1}{2}$ of $\frac{1}{4}$.
$4\frac{1}{8}$ *by* $\frac{3}{8}$ *of* $\frac{8}{5}$ *of* $\frac{9}{10}$.
$4879107\frac{2}{11}$ *by* 9.
$10009\frac{1}{121}$ *by* $\frac{1}{11}$ *of* $\frac{1}{11}$.
$91930\frac{11}{16}$ *by* $1101\frac{3}{56}$.

Addition of Decimals.

Required the sums of

29·0101 + ·046 + 221·7 + 27·9 .... *Ans.* 281·6561.

472 + 64·1 + ·0004 + 1000 .... 1536·1004.

0·9 + ·01 + ·022 + ·056 + ·00796 + ·00404 1.

What is the sum of

1 *tenth*, 39 *hundredths*, 46 *thousandths*,
229 *ten thousandths*, 7012 *hundred thousandths*,
799 *millionths*, and 179 *hundred millionths?*

Ans. 1·33132699.

Subtraction.

Required the differences

of ·501 and ·21 .... *Ans.* ·291.
2·101 1·11 .... 1·9789.
26·011 35·911 .... 9·9.
·001 1 .... 0·999.
100 ·99 .... 99·01.
20·01 20·1 .... 0·09.
·005 500
1000 ·00001
13 *thousandths* and 13 *millionths.*
1 and 1 *hundred thousandth.*

Multiplication.

Required the products

of 4·01 and ·24 .... *Ans.* ·9624.
·112 ·02 .... ·00224.
·0041 21 .... ·0861.
4·4 4·4 .... 19·36.
·042 2400 .... 100·8.
100 5·216 .... 542·6.
10000 5·426 .... 54260.
716800 ·0009765625 .... 700.
22·22 6·25 and 11·4 .... 1583·175.
5000 ·0001
6000 ·00006
1000 ·001
4·096 ·244140625
3·125, 2·048, and 15625
64, 6·4, and 5·36376953125.

Division.

Divisors.	*Dividends.*	*Quotients.*
·04	·00448	·112.
4·01	1·9248	·48.
·0082	·1722	21.
8·8	38·72	4·4.
2400	100·8	·042.
5·426	54260	10000.
2500	·0412	·00001648.
10000	7410·01	·741001.
100	·62	·0062.
·125	100	800.
700	2·25	·0032142857 &c.
3510	23·4	·00666 &c.
29100	46211·72	1·58813 &c.
1000	97400	
64	6111	
4200	56126	
288	·3456	
·288	3456	
·00288	345600	
2880	·003456	
0288	·3456	
·125	10000	
10000	·125	

Divide the sum of ·375 and ·0625 by their difference.

Divide 1400 by ·001953125.

Reduce to decimals

the fractions	$\frac{1}{3}$	*Ans.* ·3333 &c.
	$\frac{2}{3}$	·6666 &c.
	$\frac{7}{8}$	·875.
	$\frac{13}{16}$	·8125.
	[illegible]	·03125.
	[illegible]	·012557 &c.
	[illegible]	·9333 &c.
	[illegible]	·95.
	[illegible]	·[illegible]10101 &c.
	[illegible]	·2666 &c.

$\frac{1}{100}$ .... ·01.
$\frac{3}{400}$ .... ·0075.
$\frac{2197}{4096}$ .... ·536376953125.
$\frac{69}{8}$ .... 8·625.
$\frac{1}{500}$, $\frac{11}{5000}$, [illegible], [illegible], [illegible].

Duodecimals.

	f. in.		f. in.		
Multiply	7 6	by	4 8	....	*Ans.* 35 *square feet.*
	9 10	by	8 11	....	87 *sq. f.* 98 *sq. in.*
	[illegible] 5½	by	17 4½	....	511 *f.* 2 *in.* 3½ *l.*
					or 511 *sq. f.* 27½ *sq. in.*
					or 511$\frac{61}{324}$ *sq. feet.*

If the length and breadth of a board be 7 *f.* 8 *in.*, and 13½ *in.*; what is the content in square feet?

Ans. 8$\frac{5}{8}$.

The parts of the section of a field-work being

f.	in		f.	in
12	4	×	2	6
3	3	×	3	10
16	8	×	10	4
4	5	×	6	2
3	7	×	2	9

Required the whole number of square feet?

Ans. 253$\frac{23}{48}$.

Reduction.

Reduce £[illegible] to pence. *Ans.* 221280.
1000000 farthings to pounds, &c. £1041 13 4.
£19 19 10¾ to farthings. *Ans.* 19[illegible]5.
£2 [illegible]½ to farthings *Ans.* 2310.
885[illegible]5 pence to pounds, &c. £3688 1 11.
[illegible]6½ guineas to pence *Ans* 6678.
£$\frac{3}{17}$ to pence. *Ans.* 42$\frac{6}{17}$
£[illegible] to the denomination (or fraction) of a penny?
Ans. [illegible]

What is $\frac{13}{17}$ of a pound? *Ans.* [illegible]
11*d.* to the fraction of a pound. [illegible]
3½*d.* to the fraction of a pound. [illegible]

$3\frac{1}{2}d.$ to the fraction of a shilling. *Ans.* [illegible]

$10s.\ 6\frac{1}{4}d.$ to the denomination (or fraction) of a pound.
Ans. $\frac{101}{192}$.

£1 to the fraction of a guinea. *Ans.* $\frac{20}{21}$.

7 farthings to the fraction of a shilling. *Ans.* $\frac{7}{48}$.

$\frac{5}{9}$ of a guinea to shillings, &c. *Ans.* $11s.\ 8d.$

£$\frac{1}{28}$ to the denomination (or fraction) of a shilling.
Ans. $\frac{5}{7}$.

$\frac{2}{5}$ of a crown to the fraction of a guinea. *Ans.* $\frac{2}{21}$.

$\frac{1}{2}$ a guinea to the fraction of a pound. *Ans.* $\frac{21}{40}$.

$5s.\ 0\frac{3}{4}d.$ to the fraction of half a guinea.

$\frac{5}{7}$ of $6d.$ to the fraction of a shilling. [illegible]

What part of a guinea is $\frac{3}{16}$ of a pound?

What is the value of $\frac{13}{28}$ of a guinea?

Reduce £29·375 to farthings. *Ans.* 28200.

£·767 to pence. *Ans.* 184·08.

£·97 to shillings, &c. *Ans.* $19s.\ 4d.\ 3·2qr.$

42·75 shillings to pounds. *Ans.* £2·1375.

·88d to the denomination (or decimal) of a pound.
Ans. ·00366, &c.

·621$d.$ to the decimal of a shilling. *Ans.* ·052.

·25$s.$ to the decimal of a pound. *Ans.* ·0125.

3·75 farthings to the decimal of a pound. *Ans.* ·00390625.

$2s.\ 7\frac{1}{2}d.$ to the decimal of a guinea. *Ans.* ·125

$\frac{5}{7}$ of a pound and $\frac{3}{7}$ of a shilling to the decimal of a pound.
Ans. 6.

$11s.\ 10\frac{1}{4}d.$ to the decimal of a pound. *Ans.* ·5927083, &c.

·125 of a shilling to the decimal of a pound.

·019 of a penny to the decimal of a shilling.

$\frac{13}{28}$ of a guinea to the decimal of a pound.

·5082 of a penny to the decimal of a crown.

At $1s.\ 2\frac{1}{4}d.$ *per* day each man, what is the whole pay of 477 men for 365 days?

Ans. £10337 9 $8\frac{1}{4}$.

A debt of £99 18$s.$ was discharged with an equal number of $\frac{1}{2}$ guineas, crowns, and $\frac{1}{2}$ crowns; *query* the number?

Ans. 111.

Reduce [illegible] weight, to grains. *Ans.* 41400.

[illegible] to pounds, &c. *Ans.* 294$lb.$ [illegible] 15$dwt.$

[illegible] to the fraction of a $lb.$ *Ans.* [illegible]

Reduce $\frac{1}{17}$*lb.* to grains *Ans.* $338\frac{14}{17}$.
3·175*lb.* to pennyweights. *Ans* 762.
·55*lb.* to ounces, &c. *Ans.* 6*oz* 12*dwts.*
15 125*dwts* to the decimal of a *lb.* *Ans.* ·063020833 &c.

The full weight of a half-crown is 9*dwts.* $15\frac{1}{2}$*gr.* then how many are in a *lb troy?*

Ans. $24\frac{7}{...}$

Reduce $\frac{3}{7}$*lb.* (apoth weight) to ounces, &c. *Ans.* 3*oz.* 3*dr.* $1\frac{2}{7}$*sc.*
1 ton to drams, avoirdupoise weight. *Ans.* 573440.
65771*oz* to tons, &c. *Ans.* 1*t.* 16*cwt.* 78*lb.* 11*oz.*
124 drams to the fraction of a *lb.* *Ans.* $\frac{31}{64}$.
10*lb.* 8*oz.* to the fraction of a *cwt.* *Ans.* $\frac{3}{32}$.
·85*cwt* to *lbs.* &c *Ans.* 95*lb.* 3 2*oz.*
5*lb.* 4*oz.* to the decimal of a *cwt.* *Ans.* ·046875.

A cubic foot of cast iron being 461*lb.* *avoirdupoise*, then how many cubic feet are contained in a 32 pounder whose weight is 51*cwt?*

Ans. $13\frac{1}{...}$.

Suppose 20000 foot soldiers, each man having 20 rounds of cartridge with ball; now if the balls are an ounce each, and the weight of powder $\frac{1}{4}$ of the ball; what is the whole weight of lead, and of powder?

	t.	*c.*	*lb.*	
Ans.	11	3	24	lead.
	2	15	90	powder.

How many ounce, ½ ounce, ¼ and *lb.* balls, and of each an equal number, can be cast from a ton of lead?

Ans. 1280 of each.

Reduce $7\frac{1}{4}$ miles to yards, &c. *Ans.* 12906*yds.* 2*f.*
5614 feet to miles, &c. *Ans.* 10*m.* 1114*yds.*
10000 inches to yards. *Ans.* $277\frac{7}{9}$.
7 inches to the denomination, or fraction of a yard.
Ans. $\frac{7}{36}$.
$\frac{4}{5}$ of a yard to feet, &c. *Ans.* 2*f.* $1\frac{4}{5}$*in.*
$\frac{3}{7}$ of an inch to the fraction of a foot. *Ans* $\frac{1}{...}$.
$5\frac{1}{3}$ inches to the fraction of a foot. *Ans.* [illegible]
[illegible] feet to the fraction of a yard. [illegible]
$\frac{1}{17}$ of a mile to the fraction of yards. [illegible]
100 yards to the fraction of a mile. [illegible]
[illegible] feet to inches. *Ans.* [illegible]

$113\frac{3}{5}$ feet to yards. Ans. 37[illegible]

What is the value of $\frac{11}{13}$ of a mile?

of $\frac{5}{7}$ of a fathom?

of $\frac{10}{11}$ of a foot?

Reduce $7\frac{1}{7}$ fathoms to the fraction of a mile.

$7\frac{1}{7}$ feet to the fraction of a pole.

$7\frac{1}{7}$ poles or perches to feet.

$7\frac{1}{7}$ inches to the fraction of a fathom.

·64 of a mile to yards, &c. *Ans.* 1126*yds.* 1*f.* 2·4*in.*

·125 of a foot to inches. *Ans* $1\frac{1}{2}$.

10·56 miles to feet. *Ans.* 55756·8

429·85 fathoms to feet. *Ans.* 2579·1.

·855 of a foot to the decimal of a yard. *Ans.* ·285.

2·84 feet to the decimal of a yard. *Ans.* ·9466 &c.

·0095 of a foot to the decimal of an inch. *Ans.* ·114.

$10\frac{3}{4}$ inches to the decimal of a foot.

Ans. ·89583 &c

2*f.* $3\frac{1}{2}$*in.* to the decimal of a yard. *Ans.* ·76385 &c

·074418 of a fathom to the decimal of a foot.

·01356 of an inch to the decimal of a foot.

·071118 of a foot to the decimal of a fathom.

·015 of a mile to poles

·5076 of an inch to the decimal of a yard.

What is the value of ·0625 of a mile?

·7802 of a pole?

·145 of a fathom?

·124 of a yard?

·85 of a foot?

Reduce 1000 *toises* to fathoms *Ans* 106[illegible]

1000 fathoms to *toises* *Ans* 938 &c

$4\frac{1}{2}$ English miles to *toises.* *Ans* 3715·69 &c

9000 Rhynland feet to yards *Ans* 3099.

10 German miles (15 to a degree) to English miles.

Ans $46\frac{1}{15}$, nearly

The circumference of the earth being 360 degrees, and each degree $69\frac{1}{6}$ miles, what is the number of yards?

Ans. 43821000.

If the [illegible] step of a horse is $2\frac{3}{4}$ feet; then how many in a mile?

Ans. 1920.

If a company of foot march 65 *paces* of 2½ feet each in a minute; what is the rate *per* hour?

Ans 1*m* 1490*yds.*

What is the extent of a front consisting of 100 men, allowing 32 inches per man?

Ans. 61*yds.* 4*in.*

How many palisades will surround a square fort whose side is 150 yards, the centres of the palisades being 10 inches asunder?

Ans. 2160.

If I observe the flash from a cannon, and 6 seconds after hear the report, what is its distance; the velocity of sound being 1100 feet *per second?*

Ans. 2200*yds.*

Reduce	74$\frac{7}{8}$ square feet to inches.	*Ans.* 10782.
	100$\frac{7}{18}$ square yards to feet.	*Ans.* 903½.
	64212 square inches to feet.	*Ans.* 445$\frac{11}{12}$.
	119*f* 10 *in* (square) to yards.	*Ans.* 13[illegible].
	56 *sq in* to the fraction of a square foot.	*Ans.* $\frac{7}{18}$.
	·85 of a foot square to inches.	*Ans.* 122·4.
	7·48 feet *sq.* to the decimal of a yard.	*Ans.* ·8311 &c.
	59290 square yards to acres.	*Ans.* 12¼.
	7846729 square links to acres.	*Ans.* 78·46729.
	What is the value of $\frac{3}{5}$ of a square foot?	
	·85 of a square yard?	
	·755 of an acre?	
	·755 of a square pole?	

Reduce 111½ square inches to the fraction of a yard square.
·1296 of an inch square, to the decimal of a yard square.

Reduce	140¾ cubic yards to feet.	*Ans.* 3800¼.
	·56 of a cubic foot to inches.	*Ans.* 967·68.
	9816*f.* 980*in.* (*cub.*) to yards.	*Ans.* 364$\frac{[illegible]}{11664}$.
	100 bushels (*dry meas.*) to pints.	*Ans.* 6400.
	4⅓ quarters to gallons.	*Ans* 277⅓.
	2900 pecks to quarters.	*Ans.* 90[illegible].

If 1 horse is allowed 1¼ pecks of corn in 2 days, ho[illegible]ters will serve 70 horses 39 weeks?

Reduce $7\frac{3}{4}$ hogsheads *(beer meas.)* to pints. *Ans.* 3348.
64237 gallons to barrels. *Ans.* $1784\frac{13}{36}$.

How many hogsheads of beer will serve a garrison of 1350 men for 78 weeks, allowing each man $1\frac{1}{4}$ pints *per* day?

Ans 2559*h*. $20\frac{1}{4}$*gall*.

Reduce $2\frac{7}{15}$ hours to seconds *Ans.* 9120.
$\frac{3}{25}$ of a minute to the fraction of an hour *Ans* $\frac{1}{125}$.
365*d* 5*h* 48*m*. 48 sec (the solar year) to seconds.
Ans 31556928
7·96 degrees of a circle to minutes *Ans* 477·6.
25 seconds to the decimal of a degree *Ans* 0·006944 &c.
What is the value of ·0825 of a degree?
of ·625 of a minute of a degree?
of ·44 of an hour?

N. B. 60 *seconds* make a *minute*, and 60 *min* a *degree*

Compound Addition.

1. Suppose a debt is discharged in 6 weeks, after the following manner, namely, 3*l.* 17*s.* $7\frac{1}{4}$*d.* the *first* week, twice that sum the *second*, three times that sum the *third*, four times that sum the *fourth*, five times that sum the *fifth*, and six times that sum the *sixth*; what was the debt?

Ans. £81 10 $6\frac{1}{4}$.

2. What is the whole amount
of 41 guineas,
37 half guineas,
£21,
19 crowns,
33 half-crowns,
101 dollars, at 4*s.* $2\frac{1}{4}$*d* each,
147 gold mohurs, at 1*l* 13*s.* $2\frac{1}{2}$*d* each,
191 sicca rupees, at 2*s* $2\frac{1}{2}$*d.* each?

Ans. 391*l.* 0*s.* 1*d.* $2\frac{7}{10}$*qrs.*

3. What is the sum of 10*l.* $12\frac{1}{2}$*s.*—$1\frac{3}{8}$*l.*—3*l.* 8*s.* $4\frac{1}{2}$*d.*—and 17*s* $6\frac{3}{4}$*d.*?

Ans. 16*l.* 3*s.* $1\frac{13}{24}$*d.*

4. R[illegible] sum of 8·76*l.*—21*l.* 16·44*s.*—and 19*s* 10·32*d.*

Ans. 31*l.* 11*s* 6*d*

5. [illegible] whole weight of 12 barrels of gunpowder, three being 80*lb.* 15½*oz.* each, four 93*lb.* 9¾*oz.* each, and the other five 101*lb.* 11½*oz.* each?

Ans 10*cwt.* 5*lb.* 15*oz.*

6. Suppose the superficial contents of the several parts of the section of a field-work

	f.	*in.*
are	79	54
	47	111
	49	67
	64	8
	100	64
	19	13
	30	34

What is the content in square yards?

Ans. $43\frac{55}{144}$.

7. A field was measured in three divisions; the first contained 1*ac.* 143[illegible]*pol.* the second 5*ac.* 8*ch.* 4500 *links*, and the third 12530 *yards*; required the whole content?

Ans. 13*ac.* 58[illegible] *pol.*

8. If the cubic contents of the ditch surrounding an irregular pentangular work

	feet	*in.*
are	36601	614
	27720	1700
	23761	49
	35640	1606
	31681	945

What are the cubic yards?

Ans. 5755[illegible].

Compound Subtraction.

1. What is the difference of 3 guineas, and 3 times 17*s.* 10¾*d*?

Ans 9*s.* 3¾*d.*

2. Suppose a person owed 117 guineas, what would he be indebted after paying the following sums:

	£	*s*	*d.*
viz.	40	17	6½
	16	12	11¼
	10	5	9¾
	5	19	1½
	9	11	7¾

[illegible] 9*s.* 11[illegible]

3. If the discount on 80*l.* is 1*l.* 4*s.* 6$\frac{1}{3}$*d.*—on 100*l.* 10[illegible] 1*l.* 10*s.* 6*d.* 1$\frac{1}{3}$*qrs.*—on 200*l.* is 2*l.* 11*s.* 4$\frac{1}{2}$*d.*—and on 90*l.* is 17*s.* 11$\frac{1}{4}$*d.* What is the whole difference or sum to be received?

Ans 463*l* 16*s.* 7*d* 2$\frac{181}{1051}$*qrs.*

4. If the quantity of provisions in a garrison is 111*ton.* 12*cwt.* how much would be left at the expiration of 7 weeks, supposing the weekly consumption to be 12*ton.* 13*cwt.* 1*qr.* 21*lb* 7*oz* ?

Ans. 22*ton.* 17*cwt.* 3*qr.* 17*lb.* 15*oz.*

5. If three pieces whose lengths are 4*f* 10·6*in*—2*f* 7·7*in*—and 1*f.* 5·5*in.* be cut from a plank whose length is 4*yds.* 1*f* 9$\frac{1}{2}$*in.* how long is the remainder?

Ans. 1*yd.* 1*f* 9·7*in.*

6. From a piece of ground containing 3*ac* 4$\frac{1}{2}$*pol.* a part equal to 1050 square yards was marked off for a surrounding ditch. Required the content of the inner space?

Ans 2*ac.* 129$\frac{1}{2}\frac{9}{21}\frac{1}{2}$*pol.*

7. Three hogsheads and an half of liquor, wine measure, being poured into a vessel whose cubic capacity was 1*yd.* 7*f.* 13*in.*; what remained empty?

Ans. 4*f.* 917$\frac{1}{2}$*in.*

Compound Multiplication and Division.

1. When oats are at 3*s* 11$\frac{1}{4}$*d. per* bushel, what is that *per* quarter?

Ans 1*l* 11*s* 6*d*

2. What must be given for [illegible]9 sacks of barley at 1*l* 7*s* 7$\frac{3}{4}$*d per* sack?

Ans. 13*l* 16*s* 5$\frac{1}{2}$*d*

3 At 9s. 10$\frac{1}{2}$*d. per* bushel, what is that *per* load of 40 bushels?

Ans 19*l.* 15*s.*

4. At 1*s* 0$\frac{3}{4}$*d. per lb.* what cost 16 barrels of gunpowder, each weighing [illegible]

Ans. 76*l.* 10*s.*

5. What cost 29 yards of cloth at 4*s.* 5$\frac{1}{4}$*d. per* yard?

Ans. 6*l.* 8*s.* 8$\frac{1}{4}$*d.*

6. At 1s. [illegible] *per lb.* what is that *per* hundred weight?

Ans. 6*l.* 17*s.* 8*d.*

7. At 3*s.* 7¼*d.* *per* day what is that *per annum*, or for 365 days?

Ans. 65*l.* 15*s.* 6¼*d.*

8. What is the expense *per annum*, or for 365 days, of a regiment of cavalry, according to the following statement:

		£	s.	d.	
	Colonel	1	15	0	*daily pay.*
2	Lieutenant Colonels *each*	1	4	6	
2	Majors .... *each*	1	0	6	
7	Captains .... *each*	0	15	6¼	
	Captain Lieutenant	0	9	0	
10	Lieutenants .... *each*	0	9	0	
10	Cornets .... *each*	0	8	0	
	Adjutant	0	5	0	
	Chaplain	0	6	8¼	
	Surgeon	0	6	0¾	
2	Surgeon's Mates ...... *each*	0	3	6½	
	Paymaster	0	15	6½	
10	Quarter Masters .... *each*	0	5	6	
	Serjeant Major	0	2	2¼	
40	Serjeants .... *each*	0	2	2	
	Trumpet Major	0	2	2¼	
9	Trumpeters .... *each*	0	1	7	
40	Corporals .... *each*	0	1	7½	
709	Privates .... *each*	0	1	3	

Clothing.

	Serjeant Major.	0	0	6	*per day.*
40	Serjeants .... *each*	0	0	6	
	Trumpet Major	0	0	6	
9	Trumpeters .... *each*	0	0	4	
40	Corporals .... *each*	0	0	4	
709	Privates .... *each*	0	0	4	

Arms and Appointments.

	Serjeant Major	0	1	2½	*per day.*
40	Serjeants .... *each*	0	1	2½	
	Trumpet Major	0	1	0	
9	Trumpeters .... *each*	0	1	0	
40	Corporals .... *each*	0	1	2¼	
709	Privates .... *each*	0	1	2¼	

Forage.

87	Officer's Horses	*each*	0	1	5¼	*per day.*
800	Troop Horses	*each*	0	2	1¾	

Ans. £81762 17 8[illegible]

9. What is the annual expense, or for 365 days, of a regiment of foot, consisting of 10 companies; according to the following statement?

			£	s	d	
	Colonel		1	[illegible]	6	*daily pay,*
2	Lieutenant Colonels ...	*each*	0	15	11	
2	Majors	*each*	0	14	1	
7	Captains	*each*	0	9	5	
	Surgeon		0	9	5	
	Assistant Surgeon		0	5	0	
10	Lieutenants	*each*	0	5	8	
	Quarter Master		0	5	8	
10	Ensigns	*each*	0	4	8	
	Adjutant		0	5	0	
	Paymaster		0	[illegible]	0	
40	Serjeants	*each*	0	1	6[illegible]	
40	Corporals	*each*	0	1	2¼	
10	Drummers	*each*	0	1	1[illegible]	
910	Privates	*each*	0	1	0	

Clothing.

40	Serjeants	*each*	0	0	5	*per day*
40	Corporals	*each*	0	0	1	
10	Drummers	*each*	0	0	4	
910	Privates	*each*	0	0	[illegible]	

Arms and Appointments.

40	Serjeants	*each*	0	0	[illegible]	*per day*
40	Corporals	*each*	0	0	1	
10	Drummers	*each*	0	0	1	
910	Privates	*each*	[illegible]	0	1[illegible]	

Ans. £[illegible] 1 [illegible]

10. What cost 25½ quarters of oats, at [illegible] per quarter? *Ans.* 10*l* 0*s*. 0¾*d*.

11. At [illegible] 11½*d*. per yard, what cost 5[illegible] yards? *Ans* 11*l*. 15*s*. 11*d*. [illegible]

12. At $2l.$ 11s. $7\frac{1}{2}d.$ *per* hundred weight, what cost $10\frac{3}{4}cwt.$?

Ans $27l.$ 30s. $8d.$

13. What cost $53\frac{1}{2}lb.$ of powder at 1s. $0\frac{1}{4}d.$ *per lb.*?

Ans. $4l.$ 15s. $5d$ $1\frac{1}{2}$ *qrs.*

14. What is the neat weight of 38 barrels of gunpowder, the gross weight of each being $96lb.$ $14oz.$ and that of each empty barrel $8lb.$ $7oz.$?

Ans. 30*cwt.* 10*oz.*

15. What is the weight of 14 guineas, each being 5*dwts.* $9\frac{1}{4}gr.$?

Ans. 11*oz.* 17*dwts.* 10*gr.*

16. What is the whole length of 26 planks, each being 5*yds.* 2*f.* 4·7*in.*?

Ans. 150*yds.* 2*f.* 2·2*in.*

17. How many square yards are contained in 17 boards, each being 13*f.* 57·8*in*?

Ans. 25*y.* 2*f.* 118·6*in.*

18. If 1 man can dig 6*yds.* 13*f.* cubic measure in a day, how much would 57 men dig in 3 days?

Ans. $1108\frac{1}{3}yds.$

19. How many hogheads of beer in 47 barrels, each barrel containing [illegible] *gall.* [illegible] *pints*?

Ans. 30*hds* 19*gall.* 1*p.*

20. If oats are [illegible] *per* quarter, what is that *per* bushel?

Ans. 4s. $11\frac{3}{8}d$

21. When [illegible] *per* chaldron, what is the price of a bushel?

Ans. 1s. 2d. $3\frac{1}{3}$*qrs*

22. If the whole pay of 100 men be [illegible] 11s. 3*d.* for a week, what is the daily pay of each?

Ans 1s. $2\frac{1}{4}d.$

23. If I give [illegible] for 23 yards of cloth, what is that *per* yard?

Ans. 3s 10*d* $2\frac{6}{23}$*qrs*

24. If [illegible] gunpowder cost [illegible]s. $1\frac{1}{2}d$ what is that *per lb.*

[illegible] 1*d.*

25. If $3\frac{3}{4}cwt.$ cost $22l.$ 6s 3*d.* what is that per *lb.*?

Ans. 1s. $0\frac{3}{4}d.$

26. If the weekly expenditure of provisions in a garrison be 4 *ton* 17*cwt.* 50*lb.* what is that *per* day?

Ans. 13*cwt.* 103$\frac{1}{7}$*lb.*

27. If the ground for a fort contains 27*ac.* 29$\frac{1}{2}$*pol.* and $\frac{1}{5}$ is marked off for the surrounding ditch, what is the content of the remainder?

Ans. 21*ac.* 119$\frac{3}{5}$*pol.*

28. If 84 men dig 2924*yds.* 12*f.* cubic measure, in 6 days, what is that *per* day for each man?

Ans. 5 *yds.* 21$\frac{2}{3}$*f.*

29. Required the calibre, or diameter, of a cannon-ball, when it is $\frac{1}{24}$ of the length of the bore, supposing the bore to be 7*f.* 11·6 *in.*?

Ans. 3·9833 &c. *inches.*

Aliquot-Parts.

1. Required the product of 683 and 2$\frac{1}{2}$?

Ans. 1707$\frac{1}{2}$.

2. Required the product of 5167 and 3$\frac{1}{4}$?

Ans. 17767$\frac{1}{4}$.

3. What is the product of 101657 and 21$\frac{3}{4}$?

Ans. 2276289$\frac{3}{4}$.

4. What is the product of 553 and 7$\frac{2}{3}$?

Ans. 4239$\frac{2}{3}$.

5. What is the product of 98167 by 19$\frac{7}{12}$?

Ans. 1922137$\frac{1}{12}$.

6. Required the product of [illegible] and 110[illegible]?

Ans. [illegible]13215$\frac{9}{25}$.

7. What is the product of [illegible] and [illegible]$\frac{1}{2}$?

Ans. 1312$\frac{3}{4}$.

8. Required the product of [illegible]$\frac{3}{4}$ and 45[illegible]?

Ans. 668560$\frac{1}{8}$.

9. Required the product of 54*f.* by 7*f.* 6*in.*?

Ans. 630 *feet* square.

10. Let 36*f.* 6*in.* be multiplied by 10*in.*?

Product 30$\frac{5}{12}$ *feet* square.

11. What is the expense of digging a ditch 511 yards long, at 4*s.* 7$\frac{1}{2}$*d.* *per* yard?

Ans. 118*l.* 3*s.* 4$\frac{1}{2}$*d.*

Rules of Proportion.

1. Required a 3*d.* proportional to 21 and 39?

Ans. $72\frac{3}{7}$.

2. .... to ·16 and ·071?

Ans.

3. .... to $\frac{11}{16}$ and $15\frac{1}{2}$?

Ans.

4. Required a 4*th.* proportional to $2\frac{1}{2}$, $19\frac{1}{3}$, and ·0111?

Ans. ·08769.

5. .... to $\frac{7}{11}$, $\frac{49}{51}$, and $\frac{151}{663}$?

Ans.

6. .... to 1·75, 8·11, and ·095?

Ans.

7. Divide 1 into two parts having the ratio of $\frac{1}{4}$ to $\frac{1}{6}$.

Ans.

8. Let 10 be divided into three parts that shall have the same proportions as the three decimals, ·8, ·01, and ·009.

Ans.

9. If gunpowder is 1*l.* 16*s.* 6*d* *per cwt.* what cost 17*cwt* 2*qr.* 11*lb*?

Ans.

10. When oats are 1*l.* 17*s.* 8*d* *per* quarter, what cost 17*qr.* 5 *bush.* 3 *pecks*?

Ans.

11. What will $2\frac{1}{2}$ *cwt.* of gunpowder come to at the rate of 7*lb.* for 6*s.*?

Ans. 16 *guineas.*

12. If 16*cwt.* 3*qr.* 15*lb.* of lead cost 13*l.* 15*s* 11*d*, how much will 2*ton.* $17\frac{1}{2}$*cwt.* come to?

Ans. 46*l.* 19*s.* 2*d.*

13. If the clothing of 600 men cost 1288*l.* 15*s.* what will be the expense of clothing a regiment consisting of 911 men?

Ans. 1956*l.* 15*s.* $0\frac{1}{2}$*d.*

14. If a bankrupt owes 740*l.* 18*s.* and his whole property amounts to no more than 310*l.* 12*s.* what can he pay *per* £ to his creditors?

Ans. 8*s.* 4*d.* $2\frac{1342}{1453}$*qrs.*

15. When a person's annual income is 343*l.* 10*s.* 5*d.* what should be his daily expenses in order to lay by 50*l.* a year?

Ans. 16*s* 1*d.*

16. What will the tax on 529*l.* 10*s* amount to at 2*s.* 5½*d.* in the pound?

Ans. 65*l.* 1*s.* 8¼*d.*

17. If the average step of a horse be 2¾ feet, and that of a man 2½ feet, then how many men's paces are equal to 40 of a horse?

Ans. 44

18. If a garrison of 860 men have provisions for 270 days, how long will those provisions last if the garrison be reduced to 644 men?

Ans. 360$\frac{92}{161}$ *days.*

19. Two hundred and forty men having raised a certain work in 8 days: how many men would be necessary to finish a like quantity of work in 20 days?

Ans. 96

20. If 720 men when put in column of march with 8 men in front, extend 216 paces; what will be the extent if they march 9 men in front?

Ans. 192 *paces.*

21. If a certain number of workmen can throw up an entrenchment in 10 days when the day is 6 hours long; in what time would they do it when the day is 8 hours long?

Ans. 7½ *days.*

22. If the garrison of a besieged place have provisions for 12 weeks, at the rate of 18 ounces *per* day for each man, what must be the allowance if they intend to hold out 16 weeks?

Ans. 13½.

23. What length must be cut off a board that is 14½ inches wide to make a foot square?

Ans. 9$\frac{27}{29}$ *inches.*

24. How many yards of paper which is 2 feet wide, will hang a room that is 6 yards long, 5¼ broad, and 8½ feet high?

Ans. 95⅝ *yards.*

25. If the penny loaf weighs 6½*oz.* when wheat is 12*s* 6*d* per bushel; what should it weigh when the wheat is 14*s* 10*d* the bushel?

Ans. 5$\frac{85}{178}$ *oz.*

26. If a garrison of 800 men have provisions for 12 weeks at the rate of

20 ounces a day for each man; what must be the allowance to make those provisions last 20 weeks if the garrison is reduced to 700 men?

Ans. $13\frac{5}{7}$oz.

27. If the quantity of provisions in a garrison serve 1200 men 24 weeks, at the rate of 20 ounces a day for each man; how many men will the same provisions maintain 18 weeks, allowing each man 16 ounces a day?

Ans. 2000.

28. If 840 men require 5880 rations of bread for a week, how many rations will 2520 men require for 7 weeks?

Ans. 123480.

29. In the latitude of London, the distance round the earth on the parallel of latitude is nearly 15060 miles; now as the earth turns round once in 23h 56m 4sec. at what rate *per* minute is the City of London carried from west to east by this motion?

Ans $10\frac{17}{21}\frac{290}{341}$ *miles.*

30. Suppose a General imposes a contribution of 2000*l.* on 4 towns, to be paid in proportion to the number of inhabitants contained in each; now if the first contains 1200, the second 1400, the third 1600, and the fourth 1800, what part must each town pay?

Ans £ 400, $466\frac{2}{3}$, $533\frac{1}{3}$, 600.

31. Four companies, consisting of 42, 57, 66, and 78 men, respectively, being sent into a garrison where the duty requires 81 men a day; how many must each company furnish in proportion to its strength?

Ans. 14, 19, 22, and 26.

32. Suppose the forage on $2\frac{1}{2}$ acres of land will supply a body of 400 horse for 3 days; how many such acres will serve 750 horse for 7 days?

Ans. $10\frac{15}{16}$.

33. If the charge of keeping 10 horses 52 weeks is 457*l.*; what will the keep of 68 horses amount to in 21 weeks at the same rate?

Ans. 1254*l.* $19\frac{11}{13}$*s*

34. Three troops of horse rent a field for which they pay 80*l.*; the first

sent 56 horses for 12 days; the second sent 64 horses for 15 days; and the third sent 30 horses for 13 days; what must each troop pay?

Ans. 1st. 17*l.* 10*s.*
2*d.* 25*l.* 0*s.*
3*d.* 37*l.* 10*s.*

35. If the carriage of 30*cwt.* of baggage cost 1*l.* 4*s.* for 20 miles; what will the carriage of 76*cwt.* for 81 miles amount to at the same rate?

Ans. 12*l.* 15 $\frac{9}{?}$ *s.*

36. If a piece of canvas 18 Flemish ells long, and $\frac{3}{4}$ *yd.* wide, cost 18*s.* 6$\frac{3}{4}$*d*; what cost another piece of the same quality which is 63 English ells in length, and a yard wide?

Ans. 7*l.* 4*s.* 4$\frac{1}{2}$*d.*

37. Bought a silver tankard weighing 36[illegible]*oz. avoirdupois* at 5*s.* the ounce, and sold it at 5*s.* 5$\frac{1}{2}$*d* the ounce *troy*; what was gained or lost?

Ans. 10$\frac{[illegible]}{[illegible]}$*d. lost.*

38. Suppose 1$\frac{3}{4}$*cwt.* of gunpowder at 5*l.* 12*s* *per cwt.*; 2$\frac{1}{2}$*cwt* at 4*l.* 13*s.* 4*d.* *per cwt.*; and 2$\frac{1}{4}$*cwt* at 6*l.* 1*s.* 4*d* *per cwt.* to be mixed together; what is a hundred weight of the compound worth?

Ans. 5*l.* 8 $\frac{[illegible]}{[illegible]}$*s.*

39. A General having detached [illegible] of his army to take possession of two strong posts, and 750 men to watch the motions of the enemy, found that he had only [illegible] his army left; what was his whole force?

Ans. 3[illegible]00 men.

40. The ordinary Grecian army consisted of 28672 men: the *psiles* or light armed foot were twice the number of the cavalry, and the *oplites* or heavy armed foot were twice the number of the light armed. Query the number of each?

Ans. Cavalry 4096.
Light armed 8192.
Heavy armed 16384.

41. Three soldiers A, B, C, divide 3850 cartridges in the following manner, viz. A took 2 as often as B took 3, and C got 5 for every 4 which B had; what number did each get?

Ans. A 880.
B 1320.
C 1650.

42. A body of 252[illegible] troops is composed of 4 battalions; what is the strength of each, if [illegible] the first, [illegible] of the second, $\frac{1}{4}$ of the third, and $\frac{1}{5}$ of the fourth are equal?

Ans. 360, 540, 720, 900.

43. A party of foot begin their march at 8 in the morning; two hours afterwards a troop of horse follow them (from the same place): the foot march 80 paces *per* minute, and the horse [illegible]; now if a man's step be $2\frac{1}{2}$ feet, and that of a horse [illegible] feet; in what time will the horse overtake the foot; and what distance will they have marched?

Ans. 8[illegible]. 25[illegible] *min.*

Dist. 23*m.* 301[illegible] *feet.*

44. At what time between 10 and 11 o'clock are the hour and minute hands of a watch together?

Ans. $54\frac{6}{11}$*min.* [illegible] 10.

45. A party of horse leave London for Oxford at 7 in the morning; and another party leave Oxford for London at 9 the same morning; the former march $3\frac{3}{4}$ miles an hour, and the latter [illegible]; how far will each have travelled when they meet, the distance from Oxford to London being 59 miles?

m.

Ans 30[illegible] from London.

28[illegible] from Oxford.

46. A bank of earth [illegible] yards long was to have been raised by 10 men in 7 days, but at the end of [illegible] days only [illegible] yards were completed; how many men should be added to finish the bank in the proposed time at the same rate of working?

Ans. 10.

47. A General after detaching $\frac{5}{13}$ of his army to take possession of a height, and [illegible] of the remainder to reconnoitre the enemy, had 1280 men left, what was his whole force?

Ans 3380 men.

48. If a garrison of [illegible] men have provisions for 12 months, but at the end of [illegible] months are reinforced with [illegible] men, and 2 months after that with 10[illegible] more, how long will the provisions last, supposing no alteration in the daily allowance of each man?

Ans. $8\frac{1}{11}$ *months in the whole.*

49. Two labourers A and B if they work together can dig a trench

in 20 days; A can dig it himself in 34 days; in what time would B do it if he worked alone?

Ans 48$\frac{4}{7}$ *days*.

50. A can dig 32 yards of a trench in 6 days; B can dig 29 yards in 5 days; and C can dig 54 yards in 10 days; in what time would they finish 100 yards if they work together?

Ans. 6$\frac{3}{82}$ *days*.

51. If A can finish a certain number of yards of an entrenchment in 6 days of 7 hours each, and B can do 4 times as much in 15 days of 9 hours each; what is their comparative strength?

Ans. the strength of B is to that of A as 56 to 45.

52. Suppose 20 men in 15 days of 8 hours each, can dig 45 cubic yards; how many cubic yards can 25 men dig in 40 days of 10 hours long, supposing the hardness of the ground in the former case, is to that in the latter, as 9 to 11, and the strength of each of the 20 men is to that of each of the 25, as 6 to 7?

Ans. 178$\frac{41}{44}$ *yards*.

53. If 30 men in 40 hours can dig 80 cubic yards; how many men, which are stronger in the proportion of 5 to 1, would it require to dig 120 yards in 90 hours, supposing the ground in the latter case is harder than that in the former, in the ratio of 9 to 8?

Ans 18.

54. Suppose two labouring parties, one consisting of 40, the other of 50 men, and let the strength of each man of the former party be to that of each of the latter as 3 to 4; now if the 40 men can dig 100 cubic yards in 10 hours; in what time would the other party dig 180 yards if the ground in the former case is twice as hard as that in the latter?

Ans 10$\frac{4}{5}$ *hours*.

N. B. In the three last questions, the labour in digging a like number of yards, is supposed to be directly proportional to the hardness of the ground.

55. A plan of raising the siege of Brunswick, by Prince Ferdinand in 1761, has a *scale* of 300 *Rhynland roods*; the *scale* is just 2 62 *inches* in length: the plan is 18$\frac{1}{2}$ *inches* long, and 15$\frac{3}{4}$ broad; now if it be enlarged to 5 *inches* the English mile, what will be its length and breadth?

Ans. 29·3*in*. long.
25·4*in*. broad.

56. A, B, and C, can dig a trench in 4 days; A can do it by himself in 7 days, and B in 14; in what time would C finish it if he worked alone?

Ans 28 *days.*

57. A, B, and C, can do a piece of work in 10 days; B, C, and D, in 12 days; C, D, and A, in 14 days; and D, A, and B, in 16 days; in what time would each do it by himself?

A B C D

Ans. $44\frac{68}{113}$, $29\frac{23}{173}$, $23\frac{13}{109}$, $173\frac{23}{29}$ *days.*

58. Suppose a clock has three hands, and that one moves round once in a day, another once in 30 days, and the third once in 365 days; now if they are all together at any particular time, how long is it before they come together again?

Ans. 2190 *days.*

59. Divide 10 into three such parts, that when the 1*st.* is multiplied by 2, the 2*d.* by 3, and the 3*d.* by 4, the three products may be equal?

Ans. $4\frac{8}{13}$, $3\frac{1}{13}$, $2\frac{4}{13}$.

60. Let 10 be divided into 4 parts such, that when they are respectively divided by 2, 3, 4, and 5, the quotients shall be in the same proportion as 6, 7, 8, and 9?

Ans. $1\frac{1}{11}$, $1\frac{10}{11}$, $2\frac{10}{11}$, $4\frac{1}{11}$.

Questions respecting the march of Troops.

1. If the force of a battalion be 490 men, in three ranks, what is the extent of its front, the allowance for each man in front being 22 *inches* or $1\frac{5}{6}$ *feet*? (See quest. 27, art. 101.)

2. Suppose the same battalion in line of two ranks; what is the extent of its front?

Ans. $449\frac{1}{6}$ *feet.*

3. In what time would a column consisting of 7 battalions, the extent of each being 317 feet, march its own length at the ordinary rate of 75 *paces* of $2\frac{1}{2}$ *feet* each *per* minute?

Ans. $11\frac{313}{375}$ *min.*

4. In what time would a column of 11 such battalions march through a défilé $1\frac{1}{2}$ miles long at the same rate?

Ans. 60 [illegible] *min.*

5. Supposing the march is according to quick time or 108 *paces per minute*; in what time would the column pass through the defilé?

Ans. $12\frac{1}{4}$ *min.*

6. In what time would a column of horse whose extent is 896 *feet*, march through a defilé [illegible] in length, at the rate of 90 *paces per minute*, supposing the average step of a horse to be $2\frac{1}{4}$ *feet*?

Ans. 14 [illegible] *min.*

7. Suppose [illegible] battalions, the extent of each including 2 field pieces, being [illegible] feet, have to pass a defilé $1\frac{1}{2}$ miles in length; now if the column can move at the rate of 75 *paces* ([illegible] *feet* each) [illegible] mile, but the last $\frac{1}{2}$ mile being a bad road in which [illegible] to the cannon can march only 40 *paces* ($2\frac{1}{2}$ *feet* each) *per* minute, in what time will the column pass the defilé?

Ans. $111\frac{19}{275}$ *min.*

8. If in the last question the first mile is a bad road, and the $\frac{1}{2}$ mile a good one; in what time would the column march through the defilé; the other circumstances remaining the same?

Ans. 120 [illegible] *min.*

9. Suppose a column whose extent is 6000 *paces* of $2\frac{1}{2}$ *feet* each, has to pass a defilé $3\frac{1}{2}$ miles in length, and that it can march 80 *paces per* minute in the first mile, 50 in the next $\frac{1}{2}$ mile, 6[illegible] in the following $1\frac{1}{2}$ miles, and only 45 in the last $\frac{1}{2}$ mile, in what time will it clear the defilé?

Ans. 2*h.* 53 [illegible] *min.*

10. Admit the column A has a good road 6600 *paces* in length; the column B a middling road 4000 *paces* in length; and the column C a bad road 3310 *paces* in length, now if the first column march 108, the second column 75, and the third column only 50 *paces per* minute; how must the march be regulated that the heads of the columns may arrive at the same parallel together?

Ans. A *must halt* 5 [illegible] *min.*
B *must halt* 12 [illegible] *min.*

11. If in the last question it is required that the heads of the columns shall arrive at the same parallel at the expiration of $1\frac{1}{2}$ hours; how must the march be regulated?

Ans. A *must halt* 13 [illegible] *min.*
B *must halt* 21 [illegible] *min.*
C *must halt* 8 [illegible] *min.*

12. Suppose 17 battalions, the extent of each being 520 *feet*, have to pass 3 defilés; the first 1 mile, the second $1\frac{1}{4}$ miles, and the third $1\frac{1}{2}$ miles in length; how many battalions must pass through each defilé that the whole march through them may be made in the least time; and what will that time be if the rate of marching is 75 *paces* ($2\frac{1}{2}$ *feet* each) *per* minute?

Ans. 10 *through the shortest.* } the nearest
5 *through the next* } *whole* battalions.
2 *through the longest* }

And the respective times will be 55[illegible], 56[illegible], 54[illegible] *min.*

13. Suppose the same 17 battalions have to pass 2 defilés, the first being $1\frac{1}{2}$ miles, and the second 1 mile in length; now if the troops can march 108 *paces per* minute in the first defilé, and 75 in the other; how must the battalions be divided that the whole march through the defilés may be made in the least time?

Ans. 10 *battalions must march through the longest.*
7 *through the shortest.*

14. Suppose 22 battalions have to pass 3 defilés of equal extent; the first admitting of 4 men to march in front, the second of 6, and the third of 8; now if the length of a battalion (including 2 field pieces) when in column of march with 4 men in front is 660, with 6 men in front is 490, and with 8 men in front is 410 *feet*, respectively; how many battalions must pass each defilé that the whole march through them may be made in the least time; and what will that time be if the defilés are each 2 miles in extent, and the rate of marching 75 *paces* ($2\frac{1}{2}$ *feet* each) *per* minute?

Ans. 5 *battalions through the first* ... *time* 73[illegible] *min.*
8 .... *through the second* ... 77[illegible]
9 .... *through the third* ... 76[illegible]

15. If 12 battalions have to pass 2 defilés, one 2 miles, the other 1 mile in length, the former admitting 7, and the latter 4 men to march abreast, respectively; now if the length of a battalion (including 2 field pieces) is 280 *paces* of $2\frac{1}{2}$ *feet* each when 7 men march in front, and 407 *paces* when 4 men march in front; how many battalions must pass each defilé that the whole march through them may be made in the least time; and what will that time be, supposing the march is 70 *paces per* minute?

Ans. 4 *battalions through the broadest.* *time* 76[illegible] *min.*
8 .... *through the other.* .... 76[illegible] *mins.*

16. Suppose in the last example, the march through the shortest defilé is at the rate of 50 paces per minute, and that through the other 65; how must the battalions be divided, the other circumstances remaining the same?

time.

Ans. 6 *battalions must march through the longest* $90\frac{5}{65}$ *min.*
6 .... *through the other.* .... $91\frac{4}{50}$.

17. Admit 15 battalions of unequal strength have to pass 2 defilés; one a mile, the other $1\frac{1}{4}$ miles in length, each admitting of a like number of men to march in front; now if the extent of each of 9 battalions when in column of march is 480 *feet*, and the extent of each of [illegible] 6 is 620 *feet*, what number of battalions must pass each defilé that the whole march through them may be performed in the least time, at the rate of [illegible] *paces* ($2\frac{1}{2}$ *feet* each) *per minute*?

Ans 6 *of the less battalions, and 4 of the greater, must march through the shortest defilé.*
3 *of the less and* 2 *of the greater through the other.*
And the time of marching through the former 56[illegible] *min.*
[illegible] *through the* [illegible] 56[illegible] *min.*

N. B. In the foregoing questions, the fronts of the columns are supposed to enter the defilés nearly at the same time. And in reducing feet to paces, the nearest integer is usually taken.

Interest.

1. What is the simple interest of 319*l.* 12*s.* for 4 years at 4 *per cent. per annum*?

Ans 51*l* 2*s* 8 [illegible]*d.*

2. What is the simple interest of 217*l.* 15*s.* 8*d.* for $1\frac{1}{4}$ years, at $3\frac{1}{2}$ *per cent. per ann*?

Ans. [illegible]*l.* 4*s.* [illegible]

3. What is the simple interest of 279*l.* 10*s.* for 190 days at [illegible] *per cent. per ann.*?

Ans [illegible] 11[illegible]

4. What will be the amount of 251*l* 10*s.* in 5 years at 4 *per cent. per ann.* simple interest?

Ans. 3[illegible]

5. What is the discount on 200*l.* at 4 *per cent*?

Ans 8[illegible]

6. What is the discount of 200*l.* due a year hence at 4 *per cent. per ann.* simple interest?

Ans. 7*l.* 13*s.* $10\frac{2}{13}$*d.*

7. If 150*l.* become due to me at the end of $1\frac{1}{2}$ years, what should I receive immediately, discounting at the rate of 4 *per cent. per ann.* simple interest?

Ans. 141*l.* 10*s.* 2[illegible]*d.*

8. If I receive 275*l.* for 300*l.* due $2\frac{1}{2}$ years hence, what am I charged *per cent. per ann.* discount, reckoning simple interest?

Ans. $3\frac{7}{11}$*l.*

9. What is the purchase of 1000*l.* *bank annuities* at $91\frac{7}{8}$ *per cent.*?

Ans. 91[illegible]*l.* [illegible]5*s.*

10. What is the purchase of 1000*l.* *India stock* at $112\frac{3}{4}$ *per cent.*?

Ans. 11[illegible]*l.* [illegible]

11. What is the amount of 56*l.* 10*s.* in 4 years at $4\frac{1}{2}$ *per cent. per ann.* compound interest?

Ans. 67*l.* 7*s.* $6\frac{1}{2}$*d.*

12. What is the compound interest of 120*l.* for 5 years at 5 *per cent. per ann.*?

Ans. 33*l.* 3*s.* 0 9*d*

Double Position.

1. What two fractions are those whose sum is 1, and the greater divided [illegible] gives the quotient [illegible]

Ans. $\frac{1}{11}$ *and* $\frac{10}{11}$.

2. A general having detached 620 men to take possession of a strong post, and [illegible] of the remainder of his troops to watch the motions of the enemy, finds that he has [illegible] of his army left: what was his whole force?

Ans. 1040 *men.*

3. Three battalions [illegible] unequal force are in column of march; the extent of the first battalion [illegible] the extent of the second is equal to that of the first and third together, and the extent of the [illegible] is equal to that of the first and half [illegible]; what is the extent of the column?

Ans. 1723 *paces.*

4. What number is that which being added to its square shall make the sum 70?

Ans. [illegible] &c.

5. Required that number which added to its cube shall make the sum 70?

Ans. 4·040115, nearly.

Involution.

1. What is the square of 8765?

Ans. 76825225.

2. What is the cube of 8765?

Ans. 673373097125.

3. Required the cube of [illegible]

Ans. ·000[illegible]100[illegible]

4. What is the 4th power of [illegible]

Ans. [illegible]201.

5. What is the 13th power of [illegible]

Ans. 159[illegible]

6. Required the square of [illegible]

Ans. [illegible]

7. What is the 11th power of [illegible]?

Ans. [illegible]

8. What is the 5th power of 5[illegible]?

Ans. [illegible]

Extraction of Roots.

1. How many ranks are in a column consisting of [illegible] when the number of men in front are equal to the number of [illegible]

[illegible]

2. What is the square root of 3118801?

3. What is the square root of [illegible]

4. [illegible] the square root of 4609·0521 ?

Ans. [illegible]

5. What is the square root of ·000341880[illegible] ?

Ans. ·01849.

6. What is the square root of 11 ?

Ans. 3·3166248 *nearly.*

7. Required the square root of 3 ?

Ans. 1·7320508 *nearly.*

8. [illegible] the square root of $\frac{1[illegible]}{[illegible]}$?

Ans. [illegible]

9. What is the square root of $\frac{1[illegible]}{2[illegible]}$?

Ans. [illegible]

10. Required the square root of [illegible] ?

Ans.

11. What is the square root of [illegible] ?

Ans.

12. What is the square root of [illegible] ?

Ans. 142[illegible].

13. Required the square root of [illegible] ?

Ans. 1·8516402 *nearly.*

14. What is the square root of the decimal ·0183 ?

Ans. ·1352775 *nearly.*

15. Required the *4th* root of 37015056 ?

Ans. [illegible]

16. Required the cube root of 961504803 ?

Ans. [illegible]

17. What is the cube root of 193·100552 ?

Ans. 5·78.

18. Required the cube root of 51230158344 ?

Ans. 3714.

19. What is the cube root of $\frac{27}{64}$?

Ans. [illegible]

20. What is the cube root of [illegible] ?

Ans. [illegible]

21. Required the cube root of [illegible]?

22. What is the cube root of [illegible]?

23. What is the cube root of 11706[illegible]? *Ans.* 22[illegible].

24. Required the cube root of 16? *Ans.* 2·519842 nearly.

25. Required the cube root of 19[illegible]? *Ans.* 5·[illegible] nearly.

26. What is the cube root of the decimal ·014? *Ans.* ·2410142 nearly.

27. Required the cube root of ·000001? *Ans.* [illegible]

28. What is the cube root of [illegible]? *Ans.* ·961499 nearly.

29. The diameter of a 9*lb.* iron shot being 4 inches, what is the weight of a shot 6 inches in diameter? *Ans.* $30\frac{3}{8}$*lb.*

N. B. It is proved by geometry, that the cubic contents (and consequently the weights) of similar solids are directly proportional to the cubes of their like sides or diameters.

30. What is the diameter of a 48*lb.* iron shot? *Ans.* 6·99 inches.

31. What is the diameter of a 24*lb.* shot? *Ans.* 5·55 inches.

32. A lead ball whose diameter is $4\frac{1}{4}$ inches weighs 17*lb.* nearly; hence it is required to find the diameter of a musket ball whose weight is an ounce? *Ans.* ·656 of an inch.

33. If the depth of a barrel which holds 80 [illegible] be 20 inches; what is the depth of another barrel of similar dimensions which holds three times that quantity? *Ans.* 28·84 inches.

34. If a musket barrel which carries an ounce ball (·656 in. diam.) is 3 feet in length; what would be the diameter of the bore, and

length of a similar barrel for a pound ball, allowing $\frac{1}{50}$ of an inch for windage in both barrels?

Ans. diam. of bore 1·7029 *inches.*
length of 2·8 *inches.*

35. What is the 5*th* root of 25¼?

Ans. 1·91441 *nearly.*

36. Required the 6*th* root of 36?

Ans. 1·81712 *nearly.*

Arithmetical Progression.

1. If the first term of an arithmetical progression be ½, the common difference ¼, and the number of terms 50, what is the last term?

Ans. 12¾.

2. If the first and last terms of an arithmetical series be 18 and 2, and the number of terms 9, what is the common difference?

Ans. 2.

3. Required 3 arithmetical means between 1 and 2?

Ans. 1¼, 1½, 1¾.

4. If the first term of an arithmetical progression be 0, the last term 10, and the number of terms 20, what is the sum?

Ans. 100.

5. Suppose a triangular battalion to consist of 20 ranks, the first rank being 1 man, the next 4, the third 7, the fourth 10, and so on; what is its strength?

Ans. 590 *men.*

6. If a detachment marched 32½ miles at the rate of 4 miles the first hour, and 1 mile the last, in what time did they perform the journey, supposing each hour's march was successively diminished by the same distance, and what was that distance?

Ans. 13 *hours.*
And the decrease ¼*m. per hour.*

7. It is found that a heavy body near the earth's surface descends (by its own weight from rest) the space of $16\frac{1}{12}$ *feet* in the first second of time, $48\frac{1}{4}$ in the next second, $80\frac{5}{12}$ *feet* in third second, and so on, constituting a series in arithmetical progression, whose first term is $16\frac{1}{12}$, and common difference $32\frac{1}{6}$ *feet*: now according to this *law*, how far would a heavy body descend in 10 seconds?

Ans. $1608\frac{1}{3}$ *feet.*

Geometrical Progression.

1. If the first term be $1\frac{1}{2}$, the ratio or multiplier 3, and the number of terms 10, what is the last term?

Ans. $29524\frac{1}{2}$.

2. Let the first term be 9, the ratio or divisor, $1\frac{1}{2}$, and number of terms 8, what is the last term?

Ans. $\frac{128}{243}$.

3. Suppose the first term is 100, the ratio or multiplier 1·05, and the number of terms 8, what is the last term? In other words—What is the amount of 100*l.* in 7 years, at 5 *per cent per annum* compound interest?

Ans. £140·710042265625.

4. Let the extremes be 6 and 24, and number of terms 3; required the middle term? Or, what is the mean proportional between 6 and 24?

Ans. 12.

5. Required a geometrical mean between 10 and 20?

Ans. 14·142135 *nearly.*

6. If the first term is 22, last term 1305018, and the number of terms 4; what is the ratio, and the two middle terms?—Or let it be required to find 2 geometrical means between 22 and 1305018?

Ans. 39 *ratio.*

And the middle terms 858 *and* 33462.

7. Required two geometrical mean proportionals between 10 and 100?

Ans. 21·54435, 46·4159 } *nearly.*

8. Suppose the musket cartridges necessary for an army to be counted at 16 times; the first count being 3, the next 6, the third 12, the fourth 24, and so on; what is the whole number of cartridges?

Ans. 196605.

9. What would be the produce (or last crop) in 10 years from a grain of wheat, the increase or crop being constantly sown, and each grain producing yearly an ear of 10 grains, supposing 7000 grains to weigh a pound, and 60*lb.* to the bushel?

Ans. 3120761904*qrs.* $6\frac{2}{3}$*bush.*

10. Required the sum of the progression $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$, $\frac{3}{10000}$, &c. continued *ad infinitum*, (the ratio or divisor being 10, and last term 0)?

Ans. $\frac{1}{3}$.

11. What is the sum of the series $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c. continued *ad infinitum*?

Ans. 1.

12. The sum of three continued proportionals being 100, and the ratio of the first to the third as 1 to 4, what are the 3 numbers?

Ans. $14\frac{2}{7}$, $28\frac{4}{7}$, $57\frac{1}{7}$.

13. Suppose the ratio of the first to the 3*d*. as 2 to 3, required the three numbers?

Ans. 26·8475, 32·8813, 40·2712, *nearly*.

14. To divide 100 into 5 continued proportionals, the ratio of the first to the 5*th* being as 16 to 81?

Ans. $7\frac{123}{211}$, $11\frac{79}{211}$, $17\frac{13}{211}$, $25\frac{125}{211}$, $38\frac{82}{211}$.

OF LOGARITHMS.

154. LOGARITHMS are a set of numbers so contrived, that the products in multiplication, and the quotients in division, are obtained by means of addition and subtraction only.

155. Or, Logarithms are a series of numbers in arithmetical progression corresponding to another series of numbers in geometrical progression.

Thus if 1 be the first term of a geometrical progression, and 2 the ratio or multiplier, the terms will be

1, 2, 4, 8, 16, 32, 64, 128, &c.

(146) or 1^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, &c.

And the arithmetical series of indices or exponents

0, 1, 2, 3, 4, 5, 6, 7, &c.

are the logarithms of the corresponding terms of the geometrical series or powers of the ratio 2.

1, 2, 4, 8, 16, 32, 64, 128, &c. numbers.
0, 1, 2, 3, 4, 5, 6, 7, &c. logarithms.

156. Now the sums and differences of the indices or logarithms answer to the products and quotients of the corresponding terms or numbers.

Thus 2 + 3 make 5 the index or logarithm answering to 32.

(111.) And the product of 4 and 8 (the terms corresponding to 2 and 3) make 32.

Again, the difference of the indices or logarithms 7 and 4 is 3, the index or logarithm of the term or number 8.

And the quotient of the corresponding terms, or 128 divided by 16 is 8.

Therefore the products and quotients of the numbers in the geometrical progression are found by taking the sums or differences of the corresponding indices or logarithms.

157. But the indices 0, 1, 2, 3, 4, 5, 6, 7, &c. may denote the powers of any other number or ratio; consequently different ratios or geometric progressions give different systems of logarithms.

Thus if 1 be the first term, and 10 the ratio of a geometrical progression, the terms will be

1, 10, 100, 1000, 10000, 100000, &c.
or 10^0, 10^1, 10^2, 10^3, 10^4, 10^5, &c.

And the indices 0, 1, 2, 3, 4, 5, &c. are the logarithms of the corresponding terms or numbers, as before.

1, 10, 100, 1000, 10000, 100000, &c. *numbers.*
0, 1, 2, 3, 4, 5, &c. *logarithms.*

And according to this system or *scale*, the common logarithmic tables now in use, are calculated*.

158. Now 0 being the logarithm of 1; 1 the logarithm of 10; 2 the logarithm of 100; &c. it follows that the logarithm of any number between 1 and 10 will be 0 with a fraction; between 10 and 100, 1 with a fraction; between 100 and 1000, 2 with a fraction, &c.

159. It is also evident from the nature of the progressions, that if any number of geometrical mean proportionals be interposed between any two terms of the geometrical series 1, 10, 100, 1000, &c. and the like number of arithmetical means between the corresponding indices 0, 1, 2, 3, &c. that the latter will be the indices or logarithms of the former.

Thus the geometrical mean proportional between 100, and 10000 is 1000.

And the arithmetical mean between the indices 2 and 4 is 3, the logarithm of 1000.

In like manner the geometrical mean between 10 and 100 is $\sqrt{1000}$† = 31·6

---

* The invention of Logarithms is due to Lord Neper, Baron of Merchiston in Scotland, who first published the first table of these numbers in a small treatise, entitled *Mirifici Logarithmorum Canonis Descriptio.* His logarithms, however, are of that form which has since been called *hyperbolic logarithms.* The present scale or system of logarithms we owe to Mr. Henry Briggs, at that time (1614) Professor of Geometry at Gresham College.

The most useful Logarithmic tables, in most esteem at present for general use, are, Gardiner's, 4to 1742. Taylor's large 4to 1792. Tables Portatives, par Callet, (stereotype edition). Dr. Hutton's Mathematical Tables, 8vo. which contains a very complete History of Logarithms.

† $\sqrt{}$ signifies the *square root*, thus $\sqrt{16}$ is 4, $\sqrt{36}$ is 6.

And the corresponding arithmetical mean between the indices 1 and 2 is 1·5, which is the logarithm or index of the term 31·6227 &c.

Therefore the business of computing the logarithm of a given number principally consists in finding a geometrical mean or term of the series equal to, or nearly equal to, the number proposed; then its corresponding arithmetical mean or index will be the logarithm sought.

Now, by repeated extractions of the square root, such an approximate mean proportional may be found, as in the following example:

160. Let it be required to find the logarithm of 2.

First. The number 2 lies between 1 and 10;
(151) and the geometrical mean between 1 and 10 is $\sqrt{1 \times 10}$ $= 3·162278$.

And the arithmetical mean between the indices 0 and 1 (the logarithms of 1 and 10) is 0·5:
therefore the index or logarithm of 3·162278 is 0·5.

Secondly. The number 2 now lies between 1 and 3·162278;
And the geometrical mean between those numbers is $\sqrt{1 \times 3·162278}$ $= 1·778279$.

And the arithmetical mean or half the sum of the indices 0 and 0·5 (the logarithms of 1 and 3·162278) is 0·25:
therefore the logarithm of 1·778279 is 0·25.

Thirdly. The number 2 lies between 1·778279 and 3·162278;
and the geometrical mean is $\sqrt{1·778279 \times 3·162278} = 2·371374$.

And the arithmetical mean between the indices 0·25 and 0·5 is 0·375;
therefore the logarithm or index of 2·371374 is 0·375.

Fourthly. The terms next less and next greater than 2 are 1·778279 and 2·371374;
and the geom. mean is $\sqrt{1·778279 \times 2·371374} = 2·053525$.

And half the sum of the corresponding indices or logarithms 0·25 and 0·375 is 0·3125:
therefore the log. or index of 2·053525 is 0·3125.

And in this manner by constantly making use of the resulting geometrical means [illegible] greater than 2, after 22 extractions we get the term 1·[illegible], and the corresponding arithmetical mean or logarithm 0·301029[illegible] for its index. Therefore as [illegible] differs but 0·000001 from 2, we may take 0·30102[illegible] or 0·301030 (the greatest decimals) for the logarithm of 2.

[illegible] of the [illegible] logarithms were [illegible] computed. But other better and expeditious rules have since been derived from algebraic formulae, and the fluxional calculus.

161. Now from the logarithm of 2, the logarithms of 4, 8, 16, &c. the powers of 2, are obtained by multiplication.

Thus 0·301030 × 2 = 0·602060 the log. of 2^2 or 4.
0·301030 × 3 = 0·903090 the log. of 2^3 or 8
0·301030 × 4 = 1·204120 the log. of 2^4 or 16.
&c. &c.

162. And since 10 divided by 2 gives 5, if the logarithm of 2 be subtracted from the logarithm of 10, the remainder will be the logarithm of 5 (156).

Thus 1·000000 log. of 10.
0·301030 log. of 2.
0·698970 log. of 5.

163. And if the logarithm of 5 be multiplied by 2, 3, 4, &c. the products will be the logarithms of its powers; thus 0·698970 × 4 = 2·795880 the log of 5^4 or 625.

164. Hence in the common scale or system of logarithms, every number is supposed to be that power of 10 whose index is the logarithm of the number.

Thus by the foregoing operation $10^{0·301030}$ is equal to 2, nearly.
$10^{0·903090}$ equal to 8,
$10^{0·698970}$ equal to 5,
$10^{2·795880}$ equal to 625.
&c. &c.

165. The integral part of a logarithm is called its index or characteristic; thus in the logarithms 0·301030, 1·204120,

2·795880, the indices are 0, 1, 2; the other figures being decimals. And as the indices are easily supplied by the computer himself, they are commonly omitted in the tables.

166. Since the logarithm of the divisor taken from that of the dividend gives the logarithm of the quotient (162), it follows that the index of the logarithm of a proper fraction will be negative.

Thus suppose the logarithm of $\frac{10}{16}$, or the decimal ·625 is required:

10, its log.	1·000000	
16, its log.	1·201120	sub
	$-$1·795880	log. of $\frac{10}{16}$ or ·625.

In this subtraction 1 is carried to the index 1, which together make 2, then 1 *minus* 2 gives 1 *negative*, marked with the negative sign (—) in the remainder.

167. But the logarithm of an improper fraction will have a positive index, because its value is greater than 1.

Thus to find the logarithm of $\frac{25}{4}$ or 6·25.

25, its log.	1·397940	(twice the log. of 5.)
4, its log	0·602060	subtract.
	0·795880	log. of 6·25

168. Because 625 × 10 = 6250; and 625 × 100 = 62500, if we add the logarithm of 10, and 100 to that of 625, we get 3·795880 the log. of 6250, and 4·795880 the log. of 62500.

169. Hence it appears, that the logarithm of a whole number and that of a mixed number, or a fraction, consisting of the same significant figures, differ in nothing but the index which varies according to the place of the first figure.

Thus,

Numbers.	Logarithms.
62500	[illegible]
6250	3·795880
625	2·795880
62·5	1·795880
6·25	0·795880
·625	— 1·795880
·0625	— 2·795880
·00625	— 3·795880

Therefore the index or characteristic of any logarithm is always 1 *less* than the number of figures in the integral part of the natural number.

Explanation and use of the Table of Logarithms.

170. THE table contains the logarithms of the natural numbers from 1 to 10000, to 6 places of figures. The logarithms of the first 100 numbers are printed with the indices. Thus the logarithm of 8 is 0·903090: and the log. of 97 is 1·986772. The indices or characteristics of the other logarithms are to be annexed according to the value of the integral part of the number, as in *art.* 169.

171. To find the logarithm of a number consisting of 3 figures: suppose 123.

Look in the left-hand column for the number 123, then ·089905 in the next or 2d column is the decimal part of its logarithm; and as the number 123 consists of 3 integers, the index will be 2 (169); therefore 2·089905 is the logarithm of 123.

172. To find the logarithm of a number consisting of 4 figures: suppose 2157.

The two first figures of the logarithm of 215 are ·33; then under 7 at the top of the table, and in the horizontal row answering to 215 is 3850 which are the right hand figures of the

logarithm required: therefore the logarithm with its index will be 3·333850.

173. When the 4 right-hand figures of a logarithm are less than the 4 figures next preceding, it shows that the two first figures of the logarithm in the 2d. column are changed or augmented: thus the logarithm of [illegible] without the index is ·569956, but the logarithm of [illegible]315 is ·370[illegible].

174. To find the logarithm of a number consisting of 5 figures.

Take the logarithm of the four left-hand figures of the proposed number from the logarithm next greater; then say,

As 10, is to the difference, so is the 5th. figure of the number, to a 4th. number, which added to the least of the two logarithms gives the log. sought.

Let the number be 24676.

2467 .... log. [illegible]
next greater ... [illegible]

As 10 : 176 :: 6 : 105·6 the 4th. number.

2467 log. [illegible]
[illegible]

24676 log. 4·39227[illegible]

Here we suppose the differences of the logarithms to be nearly proportional to the differences of the corresponding natural numbers.

Thus the log. of 24[illegible] .... is 4·3[illegible]
of 24680 .... is 4·39231[illegible]
diff. of numbers 100 [illegible] 176 diff. of log.

Then, as 10 : 176 :: 6 : 105·6 the proportional part for 6, the whole for 10 being 176.

175. When the logarithm of a number consisting of 6 figures is required, the difference is taken for 100:

Thus to find the logarithm of 54·6347.

	54·6300	log.	1·737431
	54·6400	log.	1·737511
diff	100		80 diff.

Then, as 100 : 80 :: 47 : 37·6 the proportional part for 47.

$$\begin{array}{r} 1{\cdot}737431 \\ 38 \\ \hline 1{\cdot}737469 \end{array} \text{ log. of } 54{\cdot}6347.$$

But if the logarithms next less and next greater are in the latter part of the table, the required logarithm may err in the last figure when the natural number consists of 6 figures.

176. The logarithm of a vulgar fraction is found by subtracting the logarithm of the denominator from that of the numerator.

Thus to find the log. of $\frac{117}{147}$.

$$\begin{array}{lr} 117 \text{ log.} & 2{\cdot}068186 \\ 147 \text{ log.} & 2{\cdot}167317 \\ \hline & -1{\cdot}900869 \end{array} \text{ log. of } \tfrac{117}{147}.$$

Or the fraction may be reduced to a decimal.

177. A mixed number may be reduced to an improper fraction:

Thus to find the log. of $20\frac{3}{4}$,

$$20\tfrac{3}{4} = \tfrac{83}{4}.$$

$$\begin{array}{lr} 83 \text{ log.} & 1{\cdot}919078 \\ 4 \text{ log.} & 0{\cdot}602060 \\ \hline & 1{\cdot}317018 \end{array} \text{ log. of } 20\tfrac{3}{4}.$$

Or the fraction may be reduced to a decimal.

$20\frac{3}{4} = 20{\cdot}75$, and its log. is 1·317018 as before.

To find a number answering to a given logarithm.

178. This is only the reverse of finding the logarithm of a given number. Therefore look for the two left-hand figures of the proposed logarithm in the 2d. column, and for the

other figures on the right, and take out the corresponding number.

Thus the number answering to the log. 2·327155 is 212·4.
The number answering to the log. 4·350054 is 22390.
And the number answering to the log. $\bar{3}$·360404 is ·002293.

179. If the proposed logarithm is not found exactly in the table, take the difference of the logarithms next greater and next less, and also the difference between the given logarithm and the next less, then say

As the first of those differences,
Is to the second ;
So is 10
To the 5*th*. figure of the required number.

180. But if the number is required to 6 places of figures, make 100 the third term of the proportion. And the figures thus found when annexed to the number answering to the next less logarithm, will give the number sought.

Example Let it be required to find the number answering to the log. 2·265886?

Given log	2·265886	
next less	2·265761	the log. of 1844
	125	d.n.
next greater	2·265996	
next less	2·265761	
	235	diff.

As 235 : 125 :: 10 : 5 the 5*th*. figure; therefore the required number to 5 places is 18445.

But making the 3*d* term of the proportion 100 instead of 10,
As 235 : 125 :: 100 : 53 the 5*th*. and 6*th*. figures; and the number to 6 places is 184453.

This operation is exactly the reverse of that in *art*. 175. And it may be necessary to remark, that when the logarithms next less, and next greater fall in the latter part of the table where the differences are small, the number answering to the proposed logarithm cannot be depended upon to more than 5 places of [illegible].

Multiplication by Logarithms.

181. ADD the logarithms of the factors together, and the sum will be the logarithm of the product. (168)

Examples.

1. Required the product of 26 by 71?

26	log	1·414973
71	log.	1·851258
product 1846	log	3·266231

2. What is the product of 1·447 and 1·375?

1·447	log.	0·160469
1·375	log.	0·138303
product 1·989625	log.	0·298772

3. What is the product of ·0054 and ·95?

·0054	log.	-3·732394
·95	log	-1·977724
product ·00513	log.	-3·710118

In this addition 1 carried to the indices cancells *negative* 1; and the [illegible].

182. But to avoid the use of negative indices when one or more of the factors are decimals, multiply such factor or factors by 10, 100 or 1000, &c. so as to make the product or products whole numbers [illegible], then having added the logarithms of the products together, divide the corresponding number by the like 10, 100, or 1000, &c. for the answer.

Thus, taking the last example:

$$\cdot 0054 \times 1000 = 5\cdot 4 \text{ log. } 0\cdot 732394$$
$$\cdot 95 \times 10 = 9\cdot 5 \text{ log. } 0\cdot 977724$$
$$1\cdot 710118 \text{ the log. of } 51\cdot 3 \text{ which}$$

is evidently 1000 $\times$ 10 times too great; therefore 51·3 divided by 10,000 gives ·00513 the product as before.

Division by Logarithms.

183. SUBTRACT the logarithm of the divisor from the logarithm of the dividend, and the remainder is the logarithm of the quotient. (162)

Examples.

1. Divide 1416 by 59.

1416	log.	3·151063
59	log.	1·770852
quotient 24	log.	1·380211

2. Divide 25100 by 1997.

25100	log.	4·399674
1997	log.	3·300378
quotient 12·5688	log	1·099296

3. Divide ·04271 by ·8799.

·04271	log.	— 2·630530
·8799	log.	— 1·944433
quotient ·0485396	log	— 2·686097

184. **But if** we proceed as in the 3*d*. example of multiplication, remembering always to make the dividend greater than the divisor, the operation may be performed without the negative indices :

Thus, taking the last example ;

·04271 × 1000 = 42·71 log. 1·630530
·8799 × 10 = 8·799 log. 0·944433

But this quotient 4·85396 is 1000 times *too great* on account of the dividend, and 10 times *too little* because the divisor was multiplied by 10, therefore it must be 100 times *too great* ; consequently the quotient is ·0485396.

And in like manner we may avoid the negative index in all cases when the divisor is greater than the dividend.

To work a proportion by Logarithms.

185. SUBTRACT the logarithm of the divisor from the sum of the logarithms of the other two terms, and the remainder will evidently be the logarithm of the 4*th*. term or number sought.

Example 1. Required a 4*th.* proportional to 4628, 978, and 1798?

As 4628	log	3·665393
is to 978	log.	2·990339
so is 1798	log.	3·254790
		6·245129
		3·665393
to 379·958	log	2·579736

186. But instead of subtracting the log. of the first term, it will be found more expeditious to add its *arithmetical complement*:

Thus, 4628	log.	3·665393	
		6·334607	the *arithmetical complement.*
978	log	2·990339	
1798	log	3·254790	
379·958	log.	2·579736	as before.

The arithmetical complement of any number is the difference between that number and 1 with as many ciphers annexed as there are figures in the number; thus the arithmetical complement of 57 is 43, which is the difference of 57 and 100; and therefore adding 43 to any number, and subtracting 100 from the sum, must give the same difference as when 57 is taken from that number; for by adding 43 instead of subtracting 57, we get 100 too much.

That the log. of 4628 is taken from 10·000000, whence the sum becomes 12·579736; but as this is 10·000000 too much, the 10 is omitted in the index.

The easiest method of subtracting for the arithmetical complement is to begin at the left-hand and take each figure from 9, except the last figure on the right, which must be subtracted from 10.

Therefore in Division, instead of subtracting the logarithms of the divisors, add their arithmetical complements, and reject 10 in the sum of the indices for each arithmetical complement, and the result will be the logarithm of the quotient.

2. Required a 4*th.* proportional to the fractions $\frac{596}{1192}$, $\frac{749}{3745}$, and $\frac{8022}{1146}$?

As $\frac{596}{1192} : \frac{749}{3745} :: \frac{8022}{1146} : \frac{1192 \times 749 \times 8022}{596 \times 3745 \times 1146}$ the 4*th.* term in a compound fraction.

1192	....	log. 3·076276
749	....	log. 2·874482
8022	....	log. 3·904283
596	*arith. comp* of the	log. 7·2[illegible]
3745	*arith. comp*	log 6·4[illegible]18
1146	*arith comp*	log 6·940815
4*th*	term required 2 8	log. [illegible]

Here 3 tens or 30 is rejected in the sum of the indices for the 3 arithmetical complements; and the result is the log of 2 8, or of [illegible] which is the compound fraction reduced to its lowest terms.

For the log. of 11 is 1 04[illegible]
of 5 is 0·69[illegible]
0 4[illegible] log of [illegible]

3. Suppose the result of a proportion is the compound fraction [illegible] × [illegible]; what is its value?

817 }	*arith. comp* [illegible]	{ 7 072117
9171 }		{ [illegible]
15155	log	4 [illegible]
		[illegible]

Here 2 tens should be cancelled in the sum of the indices for the two arithmetical complements, but 17 is 3 short of 2 tens, therefore the index will be 3 with a negative sign,

thus — 3·34615.

The number, to [illegible] places, answering to the logarithm (without the index) is [illegible]; but the index [illegible] shows that it [illegible] be 3 places below 1, (169), therefore [illegible] is the value required, true to the last decimal.

4. Required a 4*th.* proportional to the three decimals 1427[illegible], 07168, and ·001278?

·14275 ×	10 =	1·4275	*arith comp*	log.	9·845124
·07168 ×	100 =	7·168	....	log	0·855204
·001278 ×	1000 =	1·278	....	log	0·106531
		6·6859		log.	0 82[illegible]

But the result 6·6859 is 100 × 1000 times *too great* on account of the multipliers, and 10 times *too little* because the divisor was increased 10 times (181), consequently it must be 100 × 100 or 10000 times *too great*; therefore 6·6859 divided by 10000 gives ·00066859 the 4*th.* proportional required.

Or, making use of the negative indices:

·11275 log.	—	$\bar{1}$·154576	
		10·845424	*arith. comp.*
·47168 log	—	$\bar{2}$·873201	
·001278 log	—	$\bar{3}$·106531	
Ans. nearly ·00066859 log	—	$\bar{4}$·825159	

In taking the arithmetical complement of the 1*st* term, the *negative* index must be added to 9 instead of subtracted.—And the sum of the indices (setting aside the 1 carried) make 6 *positive*, but 10 should be rejected in the sum on account of the arithmetical complement, therefore the index in the sum will be *negative* 4.

Involution by Logarithms.

187. MULTIPLY the logarithm of the number whose power is required by the index of the power, and the product is the logarithm of the power required. (161)

Examples.

1. What is the cube or 3*d.* power of 170?

170	log.	2·230449
		3 *times.*
Ans. 4913000	log.	6·691347

2. What is the 4*th* power of the decimal ·7867?

To avoid the negative index, multiply the decimal by 10 and divide the 4*th* power of the product by the 4*th.* power of 10.

·7867 × 10 = 7·867 log. 0·895809

3·583236 log. of 3830·3.

Which divided by 10000 (the 4*th.* power of 10) gives ·38303 the required power, true to 5 decimals.

Or thus:

$$\cdot 7867 \text{ log.} — \bar{1}.895809$$
$$4$$
Ans. $\cdot 38303 \text{ log} — \bar{1}.583236$

Here 3 carried to *negative* 1 make 1 *negative* the index

3. What is the amount of £60 in 50 years at 5 *per cent per ann.* compound interest?

It is evident from *Ex.* 1, *art.* 107, that $60 \times 1\cdot05 \times 1\cdot05 \times 1\cdot05$ &c. or $60 \times 1\cdot05^{50}$ is the amount.

1·05 log. 0·021189
50 *index*.
1·059450 log of $1\cdot05^{50}$
60 ..... log 1·778151
Amount £688·02 log. 2·837601

4. If in the last example, the interest is payable half-yearly, what would be the amount in the same time?

Here the amount of £1 in half a year will be £1·025.

Therefore $60 \times 1\cdot025^{100}$ is the amount.

1·025 log. 0·010724
100
1·072400
60 ...... log 1·778151
Amount £708·57 log. 2·850551

Evolution or Extraction of Roots by Logarithms.

188. DIVIDE the logarithm of the number whose root is required by the index denoting the root, and the quotient will be the logarithm of the root. (187)

Examples.

1. What is the square root of 7569.

Index 2) 3·879039 .... log. of 7569.
1·939519 .. .. log. of 87 the root.

2. Required the cube root of 10.

3) 1·000000 .... log. of 10.
0·333333 .... log. of 2·15413 *root* nearly.

What is the 4th root of ·38303, (see examp. 2, preceding art.).

$$4)\ \overline{1}\cdot583236 \ldots\ldots\ \text{log. of } \cdot38303$$
$$\overline{1}\cdot895509 \ldots\ldots\ \text{log. of } \cdot7867 \text{ root nearly.}$$

Here the operation is the reverse of that in the example referred to, and therefore in making the division by the exponent 4, we add 3 (the number carried in raising the power) to the index 1 so as to make the sum just divisible by 4, and the 3 is considered as so many tens added to the next figure on the right; hence the dividend will be — 4·35583236 which divided by 4 gives the log. of the root —but if the cube root were required, 2 must be added to make the sum just divisible by the exponent 3, and the dividend becomes — 3 2583236, the 3*d.* of which is — 1 861079 the log. of the 3*d.* root, &c.

Or thus, (without the negative index).

$$\cdot38303 \times 10^4 = 3830\cdot3 \quad \text{log } 4)\ 3.583236$$
$$0\ 895809 \text{ log of } 7\cdot867 \text{ which divided by}$$

10 the 4th root of 10^4 gives ·7867 the *root* as before.

4. What is the square root of the compound fraction $\frac{6421}{6547} \times \frac{9177}{9088}$.

6421	........log.	3.807603
9177	........log.	3 962701
6547	arith. comp. logs.	6 183[illegible]
9088		6 041[illegible]

$$-\overline{1}\ 997897 \text{ log. of } \cdot99517 \textit{ root} \text{ nearly.}$$

5. The diameter of a 9*lb* iron shot being 4 *inches*; then what is the diameter of a 48*lb.* ball; the weights being as the cubes of the diameters?

As 9*lb* . 4^3 :: 48*lb* : $\frac{64 \times 48}{9}$ the cube of the diameter

64	.... log.	1 806180
48	.... log	1 681241
9	*arith comp.* log	9·045757
		3) 2 533178
Diam. nearly 6·98*in*	log	0 844393

6 What is the diameter of a lead musket ball whose weight is 1 ounce? (See Examp 4, *art.* 419, vol. 2)

$$1 \times \cdot 2914 = \cdot 2914 \ldots\ldots \text{log.} \ldots 1\cdot 464490$$
$$3)\ \ \ 1\cdot 821497 \text{ log. of } \cdot 663 \text{ of an inch,}$$

the diameter nearly.

7. Required the geometrical mean proportional between 81 and 6561? (151.)

6561	log	3·816970
81	log.	1·908485
	2)	5·725455
Ans. 729	log.	2·862727

And the three terms are 81, 729, 6561.

For 81 : 729 :: 729 : 6561. But the square roots are also proportional (139); *viz.* 9 : 27 :: 27 : 81, whence 27 × 27 = 9 × 81. Therefore the mean proportional is the product of the square roots of the two extremes.

8. Required 3 mean proportionals between 81 and 6561? (150.)

6561	log.	3·816970
81	log.	1·908485
	4)	1·908485
		0·477121 log. of 3 the *ratio* or *multiplier*.

Therefore the 3 means are 81 × 3 = 243
81 × 9 = 729
81 × 27 = 2187

And the 5 terms are 81, 243, 729, 2187, 6561.

9. To find 4 geometrical means between 2 and 10.

10	log.	1·000000
2	log.	0·301030
	5)	0·698970
		0·139794 log. of the ratio or multiplier.
2	log.	0·301030
		0·440824 log. 2·7595 the 1st.
		0·139794
		0·580618 log. 3·8073 the 2d.
		0·139794
		0·720412 log. 5·2531 the 3d
		0·139794
		0·860206 log. 7·2478 the 4th.

Examples of Fractional Powers and Roots.

1. What is the $\frac{2}{3}$ power of 4096, or the cube root of the square of 4096, or the number answering to $4096^{\frac{2}{3}}$?

4096 log. 3·612[illegible]

3) 7·2[illegible] of [illegible] of 4096.

Ans. 256 log. [illegible] log. of the [illegible] root of that square.

2. What is the [illegible] of 1000?

$$\begin{array}{lr} 1000 \ldots \text{log.} & 3{\cdot}000000 \\ & {\cdot}4 \\ \hline \textit{Ans.}\ \text{nearly } 15{\cdot}85 \quad \text{log.} & 1{\cdot}200000 \end{array}$$

3. Required the $4\frac{1}{2}$ power of 0·98?

In this and similar cases, it is best to take the *power*, or the *root*, of the reciprocal of the proposed fraction, and then the reciprocal of that power, or root, will be the answer:

Thus the reciprocal of ·98 or of $\frac{98}{100}$ is $\frac{100}{98}$.

$$\begin{array}{rl} \frac{100}{98} \ldots \text{log.} \ 0{\cdot}008774 & \\ 4{\cdot}5 & \\ \hline 43870 & \\ 35096 & \\ \hline 0{\cdot}0394830 & \text{log. of the } 4\frac{1}{2} \text{ power of the reciprocal.} \\ -1{\cdot}9605170 & \text{log. of } {\cdot}9131 \text{ the required } \textit{power.} \end{array}$$

The log $-1{\cdot}960517$ is found by subtracting the log. 0·039483 from the log. of 1.

4. What is the ·079 power of ·079?

$$\begin{array}{rl} \frac{1000}{79} \ldots \text{log.} \ 1{\cdot}102373 & \\ {\cdot}079 & \\ \hline 9921357 & \\ 7716611 & \\ \hline 0{\cdot}087087467 & \\ -1{\cdot}912912533 & \text{log. of } {\cdot}8185 \text{ nearly, } \textit{Ans.} \end{array}$$

5. What is the 0·75 root of 2?

$$\begin{array}{rl} {\cdot}75)\ 0{\cdot}301030 & \ldots \text{log. of } 2. \\ \hline 0{\cdot}401373 & \ldots \text{log. of } 2.5198 \ \textit{Ans.} \end{array}$$

This however, is exactly the same thing as finding the *3d.* root of the *4th.* power of 2 (because $\cdot 75 = \frac{3}{4}$), and therefore 2·5198 is the number denoted by $2^{\frac{4}{3}}$.

6. Required the $31\frac{1}{4}$ root of 0·8?

10...log. 0·096910.

And $\frac{0\ 096910}{31\cdot25}$ = 0·003101 log. of the 31¼ root of the reciprocal.
= 1·996899 log. of ·9929 root required.

6. What is the ·65 root of ·0754?

$\frac{10000}{...}$...log. 1·122629.

And $\frac{1\cdot122629}{\cdot65}$ = 1·727121
= 2·272879 log. of ·018748 root nearly.

Other Examples.

1. Required the continued product of ..., ..., and ...?
Ans. ·000000224034

2...............the continued product of ..., ..., and ...?
Ans.

3...............the continued product of ·76141, ·01779, and 1·999?
Ans. 0·27077 nearly.

4...............the continued product of 1·593, ·007655, and ·011 3?
Ans.

5. Divide ... by 41 ...? .... Quot. 0·3111 &c.

6. Divide 3 by ⅓ of ...? ....Quotient.

7. Divide ·1863 by ·2863?... ....Quotient ·65072 nearly.

8.... ....·1875 by ·009375?.................. ... Quot.

9...............1 by 128?............................ .... Quot.

10. Required a 3d proportional to ... and ...?
Ans. ·400185 &c.

11...............a 3d. proportional to ... and ...?
Ans.

12...............a 3d. proportional to ·5977 and ·9777?
Ans.

13...............a 3d. proportional to ·9375 and ·9786?
Ans.

14. Required a 4*th*. proportional to [illegible], [illegible]331, and 15791?

Ans. ·31532

15. ....a 4*th*. proportional to [illegible], [illegible], and ·09876?

Ans.

16. Required a 4*th*. proportional to [illegible], [illegible], and [illegible]?

Ans.

17. What is the geometrical mean between ·07414 and [illegible]?

Ans. ·23445 &c.

18. ....between $\frac{113}{115}$ and $\frac{357}{1402}$?

Ans.

19. Required 2 geometrical means between 3 and 3000?

Ans.

20. Required the cube root of $\frac{1}{9999}$?

Ans. ·046417 *nearly.*

21. Required a 4*th*. proportional to the cube roots of 17, 19, and 21?

Ans.

22. What is the 100*th*. root of 10?

Ans. 1.02329 *nearly.*

23. To what power must 10 be raised to produce 700?

Ans.

24. Required the [illegible] root of [illegible]..*Ans.* ·037037 *nearly.*

25. What is the [illegible] root of [illegible]..*Ans.*

GEOMETRY.

DEFINITIONS.

1. GEOMETRY is that branch of Mathematics in which are considered the properties of lines, surfaces, and solids; and may be denominated the science of *extension* or *magnitude*, in contradistinction to Arithmetic, which is called the science of *number*.

Extension is distinguished into length, breadth, and thickness.

2. A line is length without breadth or thickness.

3. The extremities of a line are points. And the intersections of one line with another are also points.

4. A surface is that which has length and breadth only.

The bounds of a surface are lines.

5. A body or solid has three dimensions, namely, length, breadth, and thickness.

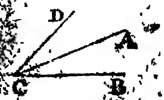
The bounds of a solid are surfaces.

6. A right line, or straight line, is that which lies all in the same direction between its extremities; and is the shortest distance between two points.

7. A plane, or plane superficies, is that in which any two points being taken, the right line between them lies wholly in that plane or superficies.

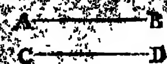
8. A rectilineal angle is the inclination of two straight lines to one another, which meet in a point called the angular point, as the angle C.

An angle is usually denoted by [illegible] the middle letter being that at the angular point. Thus, [illegible] formed by the lines AC, BC is the angle ACB. And the angle [illegible] by the lines DC, BC, is the angle DCB. Therefore [illegible] magnitude or opening of an angle is not depend[illegible] on the length of the lines which include [illegible] angle [illegible] AC, the angle DC[illegible] greater than [illegible] or the inclination of the [illegible] to BC is grea[illegible] the inclination of AC to BC.

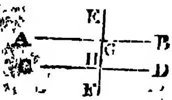


Scholium. From the foregoing definition of an angle, it follows, that if two straight lines in the same plane are not inclined to each other, they cannot form an angle, and consequently can never be produced so as to meet, in which case the lines are said to be parallel: Therefore,

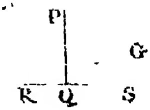
9. Parallel straight lines are such as are in the same plane but not inclined to each other, or when indefinitely produced both ways do never meet, as AB, CD.



So if two straight lines AB, CD intersect a third straight line EF (all in the same plane) and are equally inclined to that line, or make the angles AGE, CHF equal, the two lines have no inclination to one another, but are parallel or equidistant; and when all the angles at G and H are equal to each other, the line GH is the distance of these parallels.



10. A right angle is formed by two lines which are perpendicular to each other. Thus if PQ is perpendicular to RS, each of the angles PQR, PQS is a right angle.



11. An acute angle is less than a right angle; as the angle GQS.

12. An obtuse angle is greater than a right angle; as the angle GQR. Those are called oblique angles.

[illegible] The sides of a right lined plane figure are straight lines.

14. When the number of sides are three, the figure is a triangle.

15. An equilateral triangle is that whose sides are all equal, as A.

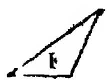
16. An isosceles triangle is that which has only two sides equal, as B.

17. A scalene triangle is when all the three sides are unequal, as C.

18. A right angled triangle is that which has one right angle, as D.

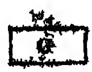
19. An acute angled triangle has all its angles acute, as E.

20. An obtuse angled triangle has one obtuse angle, as F.



21. Every plane figure bounded by four right lines is called a quadrangle or quadrilateral. And when the opposite sides are respectively parallel, the quadrilateral is called a parallelogram.

22. A rectangle is a parallelogram having all its angles right ones, as G.

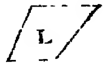


23. A square is a parallelogram having all its sides equal, and all its angles right ones, as H.

24. A rhomboid is an oblique angled parallelogram, as I.

25. A rhombus is an equilateral rhomboid, as K.

26. A trapezoid is a quadrilateral with only two parallel sides, as L.



27. A trapezium is a quadrilateral in which none of the sides are parallel, as M.

28. A right line joining any two opposite angles of a quadrilateral is called a diagonal, as NO.

29. The side PQ upon which any parallelogram PQRS, or triangle PSQ, is supposed to stand, is called the base; and the perpendicular ST, falling thereon from the opposite angle at S, is called the height or altitude of the parallelogram or triangle.

The perpendicular ST is also called the distance of the point S from the line PQ, or the distance of the parallels SR, PQ.

30. All right lined plane figures having more than four sides are generally called polygons. And a regular polygon is one whose angles as well as sides are all equal.

Axioms.

31. Things which are equal to the same thing, or to equal things, are equal to each other.

32. If equals are added to equals, the wholes are equal.

33. If equals are subtracted from equals, the remainders are equal.

34. Every whole is equal to all its parts taken together.

35. Things which are the like parts of the same thing, are equal.

36. Magnitudes which coincide with one another, that is which exactly fill the same space, are identical, or mutually equal in all their parts.

37. All right angles are equal to one another.

N. B. A Proposition is something either proposed to be done, or to be demonstrated, and is either a problem or a theorem.

A Problem is something proposed to be done.

A Theorem is something proposed to be demonstrated.

A Corollary is a consequent truth gained from some preceding truth or demonstration.

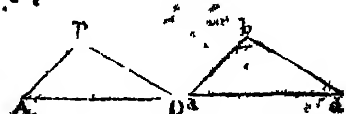
A Scholium is a remark or observation made upon something going before it.

A Lemma is something premised or demonstrated, in order to render what follows more easy.

Of the Angles of Right-lined Plane Figures.

Theorems.

1. *If in the two triangles* ABD, abd, *having two sides* BA, BD *of one triangle, respectively equal to two sides* ba, bd *of the other, and the included angles* B *and* b *also equal; the triangles are identical or equal, in all respects.*



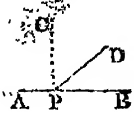
If we conceive the triangle ABD to be so applied to the triangle abd that the angle B may coincide with the angle b, and the side BA fall upon ba: Then the angles at B and b be-

ing supposed equal, the side BD will fall upon *bd*, and the point D on *d*; consequently AD will coincide with *ad*: hence it is manifest that the triangles are identical or equal in all respects; and therefore AD will be equal to *ad*, and the adjacent angles A, and D, equal to the angles *a*, and *d*, respectively.

And in a similar manner it is proved that *triangles are identical when the bases* (AD, ad) *and the adjacent angles* (A, D; a, d) *are equal.*—For if one triangle is supposed to be placed upon the other so that the bases, and adjacent angles coincide, the other sides, and also the two vertical angles, must coincide, and will therefore be respectively equal.

38. *The angles which one right line make with another on the same side, are together equal to two right angles.*

Let the line DP meet the line AB in the point P, then the two angles DPB, DPA are together equal to two right angles.



If the angles are equal, each will be equal to a right angle (10).

But when they are unequal, let PC be perpendicular to AB.

Then the three angles BPD, DPC, CPA, together are equal to two right angles (31).

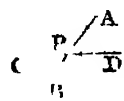
But the two angles DPC, CPA are together equal to the angle DPA.

Therefore the two angles DPB, DPA together make two right angles.

Corol. 1. Hence it appears, that all the angles at the same point (P) on the same side of a right line (AB) are together equal to two right angles. And consequently all the angles that can be made round a given point (P) are equal to four right ones.

Cor. 2. And if two angles DPB DPA on both sides of the line DP are together equal to two right angles, then the sides PB, PA make one continued line.

39. *If two right lines intersect each other, the opposite angles will be equal.*



Let AB intersect CD in the point P. Then will the angle APD be equal to the angle BPC and the angle APC equal to the angle BPD.

This might have been admitted as an axiom [illegible] all the parts of a right line lie in the same direction, the segments PB, [illegible] have the same [illegible] direction to one another [illegible] segments PC, PA on the [illegible] side of the point of intersection; [illegible] equal [illegible]

however, usually [illegible]

[illegible] The angles APC APD [illegible] and [illegible] PC together [illegible] angle APC [illegible] APD [illegible] CPB [illegible]

[illegible]

[illegible] AP, [illegible]

PB PC, [illegible] AD, [illegible]

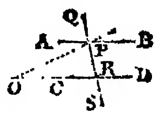
Because the side [illegible]

PBC, [illegible] to the [illegible]

of the [illegible] PAD, [illegible] P also equal (31), the [illegible] and therefore the angles opposite the equal [illegible] equal, that is, the angle BAD = PC [illegible] greater the angle PAD, [illegible] PBC

to ABC. In the same manner, if we produce BA, and bisect AC, it may be proved that the angle CAR or its equal BAG is greater than ACB.

40. *If two straight lines in the same plane intersect another straight line, and make the alternate angles equal, the two lines are parallel.*



Let the lines AB, CD, intersect QS, and make the alternate angles APS, QRD equal to each other; then AB is parallel to CD.

For if they be not parallel, the lines AB, CD are inclined to one another, and will meet when produced. Let O be the point of concourse; then RPO is a triangle, and the exterior angle PRD or QRD is greater than the interior opposite angle OPR (39th), but it is also equal to it (by construction), which is impossible, therefore the lines when produced do not meet on that side of QS: and in the same manner it may be proved that they cannot meet when produced on the other side. Therefore the lines are parallel.

And the converse is equally obvious, namely.—*If a straight line (QS) [illegible] two parallel lines* (AB. CD), *the alternate angles* APS, QSD *are equal.*

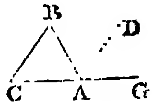
Corol. 1. Two parallel lines cannot be drawn from the same point.

Corol. 2. Because the angles APS, SPB together are equal to two right angles, and QRD, DRS also equal to two right angles, the two angles BRP, DRP together will make two right angles; therefore if two straight lines (BP, DR) in the same plane, meet another straight line (QS) and make the two inward angles (BPR, DRP) together equal to two right angles, those two lines are parallel.

Corol. 3. Hence also, if a straight line falls upon one of

several parallel straight lines, in given angles, it will intersect the other lines in the same angles.

41. *If one side of a triangle be produced, the exterior or outward angle, will be equal to both the interior opposite angles: and the three interior angles of the triangle are together equal to two right ones.*



Let the side CA of the triangle CBA be produced to G. Then the exterior angle GAB will be equal to both the interior opposite angles ABC, ACB; and the angles ABC, ACB, CAB, together make two right angles.

Draw AD parallel to CB.

Then because AD is parallel to CB, the angle DAG is equal to the angle ACB (40).

And the angle DAB is equal to the angle ABC (40).

But the two angles DAC, DAB together constitute the outward angle GAB.

Therefore [illegible] equal to both the angles ABC, ACB.

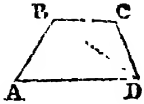
And since the three angles DAC, DAB, BAC, together make two right angles [illegible] and are respectively the same as the three [illegible] triangle CBA; therefore the sum of the three [illegible] triangle is equal to two right angles.

Corol. 1. [illegible] difference between an exterior angle of a triangle and [illegible] interior opposite angles, is equal to the other interior opposite angle.

Corol. 2. Hence also, if one angle of a triangle be a right angle, the sum of the other two make a right one.

42. *The four inward angles of every right lined quadrilateral are together equal to four right angles.*

Let ABCD be a quadrilateral. Then the sum of the angles at A, B, C, D, will be equal to four right angles.



Draw the diagonal BD, which will divide the quadrilateral into two triangles BCD, BAD.

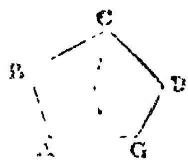
Then because the angles of those two triangles make up the four angles of the quadrilateral, and the sum of the angles of both the triangles are equal to four right angles (41), therefore the angles of the quadrilateral are together equal to four right angles.

Corol. Hence if two angles of a quadrilateral make two right angles, the sum of the other two will also be equal to two right angles.

43 *The sum of all the interior angles of any polygon is equal to twice as many right angles, wanting four, as the figure has sides.*

Let ABCDG be a polygon of 5 sid[illegible]

[illegible]en the sum of the angles at [illegible] C [illegible], will be equal to [illegible]



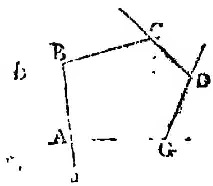
From any point P [illegible] lines [illegible] drawn to the angles of the figure, which will divide it into as many triangles as the figure has sides.

Now all the angles of the triangles are together equal to twice as many right angles as there are triangles, or as the polygon has sides.

But the angles of the triangles, exclusive of the angles at P, which make four right angles (41), constitute the interior angles of the polygon, and therefore those angles together are equal to twice as many right angles, wanting four, as the polygon has sides.

44. *The sum of the exterior angles (aAG, bBA, &c.) of any polygon, are equal to four right angles.*

Since the interior and exterior angles at each angular point of the polygon make two right angles (38, corol. 1), all the interior and exterior angles must together make twice as many right angles as the figure has angles or sides.

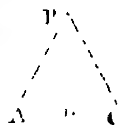


But the sum of all the inter[illegible] angles are equal [illegible] as many right angles, wanting four, as the figure has sides ([illegible]).

Therefore the difference of those [illegible], or four right angles, is the sum of the exterior angles.

46. *The angles opposite the equ[illegible] of an isosceles triangle are also equal.*

If ABC be an isosceles triang[illegible], having the side BA equal to the side BC. Then the angles at A and C are equal.



Suppose the ang[illegible] by the line BP. Then because BA [illegible] BC, and [illegible] the angle CBP, and the side BP common to both the triangles APB, CBP, those triangles will therefore be [illegible] equal in all respects (38) and con[illegible] will have the angles at A and C [illegible].

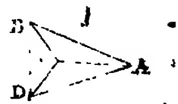
Corol. 1. Hence the line (BP) which [illegible] the vertical angle (ABC) of an [illegible] triangle, bisects the base (AC), and is also perpendicular to it.

Corol. 2. And if two angles of a triangle be equal, the sides subtending those angles will [illegible] equal.

Corol. 3. Hence also, every equilateral triangle is likewise equiangular.

[illegible] *If the sides of one triangle* (ACB) *be equal to the sides of another triangle* (ACD), *each to each; the angles opposite the like sides are also respectively equal.*

The truth of this seems sufficiently evident from Art. 38. It is however demonstrated thus:



Let a side AC of one triangle coincide with the equal side AC of the other; then AB = AD, and CB = CD.

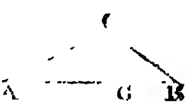
Draw BD. Then because AB = AD, and CB = CD, the triangles ABD and CBD are isosceles, and the angle ABD = ADB, and the angle CBD = CDB (46):

Now if the equal angles CBD, CDB are taken from the equal angles ABD, ADB, the two remainders or the angles ABC, ADC, must also be equal (33).

Therefore the sides CB, BA, and the included angle of one triangle, being respectively equal to the sides CD, DA, and the included angle of the other, the two triangles [illegible] identical [illegible], therefore the angle DCA [illegible] BCA, and the angle BAC [illegible] [illegible]C.

[illegible] *In any triangle* (ABC) *the* [illegible] *angle* (ACB) [illegible] *opposite* [illegible] *side* (AB).

Make [illegible] AC, and draw CG. Then because AG [illegible] AC [illegible] triangle GAC is isosceles, and the angles ACG, AGC are equal [illegible]:



But the exterior angle AGC of the triangle GBC is equal to both the angles GBC, GCB [illegible]:

Therefore the angle ACG (equal to AGC) which is only a part of the angle ACB, exceeds the angle B; consequently the whole angle ACB is greater than B.

Corol. Hence the longest side of a triangle is opposite the

greatest angle; for it is proved that ACB cannot be greater than B, except AB is longer than AC.

48. *The shortest line which can be drawn from a given point* (P) *to an indefinite line* (AB) *is that right line* (PD) *which is perpendicular to it.*

Suppose PD is perpendicular to AB: then any other line, as PR, drawn from P to meet AB will be longer than PD.

For the right angle RDP of the triangle RDP is greater than the angle PRD, because the latter with the angle P are together equal to a right angle (41, *corol.*), therefore PD is less than PR (47, *corol.*).

Of the CIRCLE.

Definitions

49. A CIRCLE is a plane figure bounded by one curve line called its circumference, which is every where equally distant from a point within it called the centre.

50. The radius of a circle is the distance of the centre from the circumference. Thus if C be the centre, CR is the radius.

51. The diameter of a circle is a right line drawn through the centre, and terminated by the circumference both ways, and therefore it is twice the radius.

52. An arc of a circle is any part of the circumference.

53. The chord or subtense of an arc AGB, or ARB, is a right line AB joining the extremities of that arc.

54. A segment is any part of a circle bounded by an arc and its chord; as the segment ABG, or ABR.

55. A semicircle is half the circle, or a segment cut off by a diameter. Half the circumference is sometimes called a semi-[illegible]

[illegible] by an arc [illegible] two radii drawn [illegible]

Thus if C be the centre, ACB is a [illegible]

When the angle at C is right, the sector (and sometimes the arc AB) is called a quadrant.

57. When two right lines AC, BC, are drawn from the extremity of a chord AB, and meet any where in the arc ADB, the angle ACB (at the circumference) is said to be in the segment ADB, and to stand on the chord AB, or on the arc ARB.

58. A right line is said to touch a circle, when it passes through a point in the circumference without cutting off any part of the circle.

This line is also called a tangent to the circle.

59. A secant is a right line which intersects the circumference of a circle.

60. Two circles are said to touch each other when the circumferences of both pass through the same point without intersecting each other.

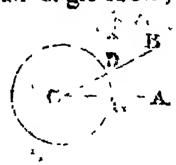
61. When all the angular points of a right-lined figure [illegible] the circumference of a circle, it is said to [illegible] circle; and the circle is said to circumscribe the figure.

62. A right-lined figure circumscribes [illegible] when all its

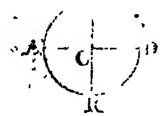
... in the circumference of the circle ... to be inscribed in the figure.

60. The perimeter of a figure is the sum of all its sides taken together.

61. When two right lines AC, BC, form an angle ACB, and a circle is described about the angular point C as the centre, the arc GD intercepted by those lines is the measure of the angle ACB, the whole circumference of the circle being the measure of four right angles.



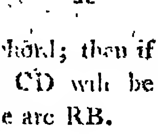
To estimate the opening or magnitude of an angle, the circumference of the circle is supposed to be divided into 360 equal parts called degrees, and each of those degrees into 60 equal parts called minutes, and each minute into 60 seconds, &c. ... Thus if the circumference ABDR is divided into 360 equal parts or degrees, and the diameters AD, BR intersect each other at right angles, the points A, B, D, R, will divide the circumference into 4 equal arcs of 90 degrees each; and each of the angles at the centre C is said to be an angle of 90 degrees.



If the arc DB is ... the whole circumference ... angle DCB will be ... degrees.

THEOREMS.

... *radius of a circle bisects a chord ... will be at right angles to it; and the arc of that chord will also be bisected by the same radius.*



Let C be the centre of the circle, and AB a chord; then if the radius CR bisects the chord in the point D, CD will be perpendicular to AB; and the arc AR equal to the arc RB.

Draw CA and CB. Then because CA is equal to CB, the

... ACR is a right angle, and therefore (16, cor.) CR bisects the angle ACB, and is perpendicular to AB.

And because the arcs AR, BR are the measures of the equal angles ACR, BCR (61), they must therefore be equal to each other.

Corol. Hence a right line which bisects any chord at right angles, will pass through the centre of the circle.

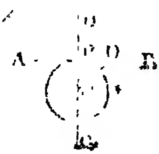
63. *In a circle, equal chords are equally distant from the centre.*

Let AB, GD be two equal chords in the circle whose centre is C; then the perpendiculars CR, CS drawn from the centre C will be equal.

Draw the radii CB, CA, CD, CG: then those radii being equal, and BA equal to GD, the triangles BCA, GCD will be identical, or equal in all respects (16); and because they are isosceles, the perpendiculars CR, CS will bisect BA, GD (16, cor.): hence the triangles RCB, RCA, SCD, SCG are identical, therefore CR = CS.

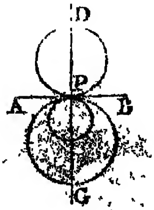
Cor. Chords in a circle equally distant from the centre are equal to each other.

64. *If two right lines AB, DG cut each other at right angles in P; and if any circle, whose centre C is in the line DG, be described through the point of intersection P, it will touch the other line AB in that point.*



Draw CO to any point in PA. Then CO being greater than CP (45), the point O must necessarily fall without the circle; and as the same reasoning holds good with respect to every other point in PB or PA, it is evident that AB cuts off no part of the circle, but touches it at P.

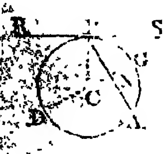
Corol. 1. Hence the angle formed by a tangent to a circle and the radius drawn to the point of contact, is a right angle.



Corol. 2. Hence also, it appears that any number of circles described through P, will touch each other in that point if their centres are in the line DG. And that AB is a tangent to them all.

Corol. 3. Therefore if two circles touch inwardly or outwardly, their centres and the point of contact are in the same right line.

68. *The angle formed by a tangent and a chord drawn from the point of contact, is measured by half the arc of that chord.*



Let RS be a tangent to the circle whose centre is C; and PA a chord drawn from the point of contact P. Then the measure of the angle SPA is half the arc PGA; and the measure of the angle RPA is half the arc PDA: that is, if a circle were described about the centre P with the radius CP or CG, the arc intercepted by PS and PA would be equal to half the arc PGA, and the arc intercepted by PR and PA equal to half the arc PDA.

For let the diameter DCG be drawn to bisect the chord PA, and join CP.

Then CNP is a right angle (64), and the angles RPC, SPC are also both right angles (67, *corol.* 1).

Now in the right angled triangle CNP, the sum of the two acute angles NCP, CPN, is equal to a right angle (41, *corol.* 2).

But the latter angle CPN together with the angle APS also make a right angle CPS.

Therefore the angle APS is equal to NCP (23). And since the arc PG (half of PGA) is the measure of the angle PCN, it must also be the measure of its equal APS.

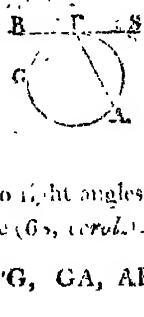
Again, the external angle DCP of the triangle CNP is equal to both the inward opposite angles, or to the angle CPN and a right angle CNP (41).

And the angle RPA is also equal to the same angle CPN and a right angle RPC.

Therefore the angles RPN, DCP are equal. And since the arc PD (half of PDA) is the measure of the angle DCP, it is also the measure of its equal RPA.

Corol. Because the arcs GP, PD together make half the circumference, and the sum of the two angles RPA, SPA equal to two right angles, therefore the sum of two right angles is measured by half the circumference.

69. *The angle at the circumference of a circle is measured by half the arc that subtends it.*



Let GPA be an angle at the circumference. Then half the arc GA is the measure of that angle.

Suppose RS is a tangent to the circle at P.

Then the sum of the three angles at P, or two right angles, is measured by half the circumference of the circle (68, corol.).

But half the circumference is half the arcs PG, GA, AP added together.

Now the angle RPG is measured by half the arc PG: and the angle SPA by half the arc AP (68):

Take those two angles from the three angles at P, and there remains the angle GPA.

And take the measures of those two angles, or half the arcs

PG, AP, from half the circumference, and there remain half the arc GA for the measure of the remaining angle GPA.

70. *All angles in the same segment of a circle, or standing on the same arc, are equal to each other.*

Let CSA, GPA be two angles standing on the same arc GA. Then will those angles be equal to each other.

For each of those angles is measured by half the arc GA (69), and consequently they must be equal.

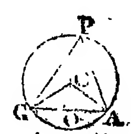
Corol. Hence equal chords in a circle, subtend equal angles at the circumference.

71. *The angle at the centre of a circle is double the angle at the circumference when both of them stand on the same arc.*

Let GAC be an angle at the centre, and GPA an angle at the circumference. Then the angle GCA is double the angle GPA.

For GCA is measured by the arc GA; and the angle GPA is measured by half that arc (69), therefore the angle GCA must be double GPA.

Otherwise thus:



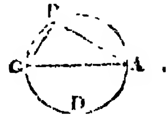
Let PO be drawn through the centre C.

Then the triangles GCP, ACP being isosceles, the angle CGP will be equal to the angle CPG; and the angle CAP equal to CPA ([illegible]).

And because the external angle GCO is equal to both the inward opposite angles CGP, CGP (41), it is therefore equal to double the angle CPG. And for the same reason, the external angle ACO is double the angle CPA: therefore the whole angle GCA is double the whole angle GPA.

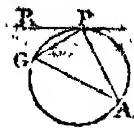
72. *The angle* GPA *in a semicircle is a right angle.*

For it is measured by half the arc GDA or half a semicircle (69), but half a semicircle is the measure of a right angle (61).



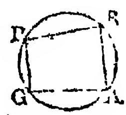
73. *The angle* RPG *formed by the tangent* RP *and the chord* PG, *is equal to the alternate angle* PAG *standing on the same chord* PG.

For the angle RPG is measured by half the arc PG (68); and the angle PAG is measured by half the same arc (69); therefore those angles must be equal.



74. *The opposite angles of any quadrangle inscribed in a circle are together equal to two right angles.*

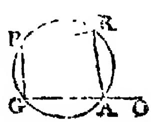
For the angle P is measured by half the arc GAR; and the angle A by half the arc GPR; therefore the sum of both angles must be measured by half the sum of both arcs, or by half the circumference.



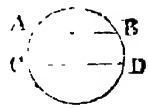
But half the circumference is the measure of two right angles, consequently the opposite angles together are equal to two right angles.

75. *If a side* GA *of a quadrangle inscribed in a circle be produced, the exterior angle* OAR *will be equal to the inward opposite angle* GPR.

For the angle GAR with its opposite angle GPR together make two right angles (74); and the same angle GAR with the exterior angle OAR make two right angles; therefore by equal subtraction, the angle OAR is eqral to the angle GPR.



76. *In a circle, two parallel chords* AB, CD *intercept equal arcs* AC, BD.



Join AD. Then because AB is parallel to CD, the alternate angles BAD, CDA, are equal to each other (40); and therefore the arc BD is equal to the arc AC (70. *corol.*).

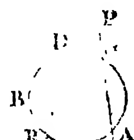
77. *The angle formed by two chords* AB, CD, *intersecting each other in a circle, is measured by half the sum of the intercepted arcs* AC, DB.

Let CR be parallel to AB.

Then the angle of intersection DPB is equal to the angle DCR (40), which is measured by half the arc DBR (69).

But the arc BR is equal to the arc AC (76); therefore the arc DBR is equal to both the intercepted arcs DB, AC; consequently the angle DCR, or its equal DPB, is measured by half the sum of those arcs.

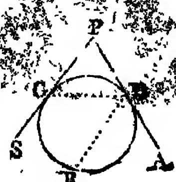
78. *The angle* P *without a circle, formed by two secants* PB, PA, *is measured by half the difference of the intercepted arcs* DC, BA.



Let CR be parallel to PB.

Then because DC [redacted] equal to BR (76), the difference of [redacted]pted arcs DC, BRA is the arc RA, half of which is the measure of the angle RCA, or of its equal BPA.

79. *The angle* P *formed by the two tangents* PS, PA, *is measured by half the difference of the two intercepted arcs* GD, GRD.



Join the points of contact C, D; and let DR be parallel to PS.

Then because DR is parallel to GP, the angle GDR is equal to DGP (40).

Now the angle DGP is measured by half the arc GD (69), and the angle GER by half the arc GR (69); therefore the arcs GD, GP are equal: consequently the arc RD is the difference of the intercepted arcs GD, and GRD.

But half the arc RD is the measure of the angle RDA (68), and therefore the measure of its equal SPA.

Corol. 1. From this and the preceding theorem, it appears, that the angle formed by the intersection of a tangent and a secant is also measured by half the difference of the two intercepted arcs.

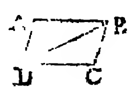
Corol. 2. Because each of the angles PGD, PDG, is measured by half the arc GD (69), those angles are equal, therefore PG = PD; hence the tangents drawn to a circle from any point without it, are equal to each other.

OF THE

EQUALITY OF PARALLELOGRAMS AND TRIANGLES.

THEOREMS.

80. *The diagonal* DB *of a parallelogram* ABCD *divides it into two equal parts or triangles* DBA, DCB.



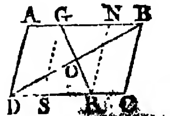
For the angles of the two triangles DAB, DCB being respectively equal, each to each (40), and the side DB common to both triangles, those triangles will therefore be identical or equal in all respects (38).

Corol. Hence the opposite sides of a parallelogram are respectively equal to each other.

81. *If a right line* GR *bisects or divides the diagonal* DB *of the parallelogram* DABC *into two equal parts in* O, *it will*

also divide the parallelogram into two equal parts or trapezoids DAGR, BCRG.

Let GS and NR be parallel to the sides AD, BC.



Then because the triangles GBO, RDO are equiangular (40), and the side OD equal to the side OB, those tri[illegible] equal in [illegible] con-sequently GB [illegible] GA = RC; hence the parallelogram [illegible] is equal to the parallelogram NBCR.

But [illegible]R divides the parallelogram SGNR into two equal triangles GSR, GNR [illegible]; therefore as the parallelograms GADS, NBCR are equal, the trapezoids GADR, BCRG, each consisting of two equal figures, must also be equal.

Corol. 1. Because DC [illegible] equal to DR and AG together, a trapezoid (DAGR) is half a parallelogram whose base is the sum of the parallel sides of the trapezoid, and whose height is the distance of those parallel sides.

Corol. 2. Hence all right lines that bisect the diagonal of a parallelogram, and are terminated by the sides, are also bisected by the diagonal.

82. *Parallelograms standing upon the same base, and between the same parallels (or having equal altitudes), are equal to each other.*

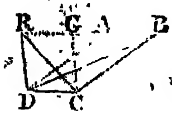
Let RB be parallel to DC. Then the parallelogram DRGC is equal to the parallelogram DABC.

For DR is equal to [illegible] and DA to CB; and RG, DC, AB equal to each other [illegible]), hence if GA be added to RG, and AB respectively, [illegible]A will be equal to GB; [illegible] therefore the sides of the triangles DRA, CGB are respectively equal; and consequently the triangles themselves must also be equal.

Now the triangle DRA being taken from the quadrilateral DRBC, the remainder is the parallelogram DABC; and if the triangle CGB be taken from the same quadrilateral there remains the parallelogram DRGC: therefore the triangles being equal, the remaining parallelograms must also be equal.

Corol. Hence it appears, that parallelograms standing upon equal bases, and having equal altitudes, are equal to each other. For if one figure be applied with its base upon the base of the other, the two parallelograms will stand on the same base, and have equal altitudes.

82. *Triangles standing upon the same base and between the same parallels (or having equal altitudes), are equal to each other.*



Let RB be parallel to DC: then the triangles DRC, DBC are equal.

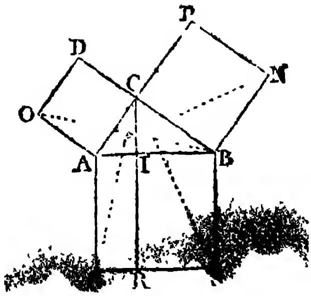
Draw CG, DA parallel to DR, CB respectively. Then the triangle DRC is half the parallelogram DRGC, and the triangle DBC half the parallelogram DABC (80): but the two parallelograms are equal (82): therefore their halves or the two triangles must also be equal.

Corol. 1. A triangle is half a parallelogram of the same base and altitude.

Corol 2. And triangles having equal bases, and altitudes are equal. For if one triangle be applied with its base upon the base of the other, the two triangles will stand on the same base, and have equal altitudes.

83. *If* ACB *be a right angled triangle; then the square* ABSC *upon the longest side or hypotenuse* AB, *is equal to both the squares* ACDO, BCPN, *upon the other two sides* AC, BC.

Draw CR parallel to AG; and join OB and CG.



Because the angles OAC, BAG are right angles, if to each be added the angle CAB, the angles OAB, CAG of the triangles OAB, CAG will be equal to each other.

And since the sides AO, AB; AC, AG including those equal angles, are respectively equal, the triangle OAB is equal to the triangle CAG (38).

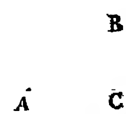
And because BD is parallel to AO, and CR to AG, the triangle AOB is equal to half the parallelogram AODC; and the triangle ACG equal to half the parallelogram AGRI (82*a*. *corol.* 1); therefore the halves being equal, the wholes must also be equal, or the parallelogram or square AODC equal to the parallelogram AGRI.

And exactly in the same manner it is proved that the triangle BNA is equal to the triangle BCS; and the square BNPC equal to the remaining parallelogram BSRI.

Corol. Hence the difference between the square of the hypotenuse and the square of either of the other sides, is equal to the square of the remaining side.

Therefore when the lengths of two sides of a right angled triangle are given, the third side may be found by extracting the square root.

Let AC = 4, and BC = 3: Then the square of AC is 16; and the square of BC is 9; and the sum of those squares is 25 the square of AB, and the square root of that square is 5, the length of AB.

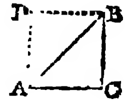


Again, suppose AB = 10, and BC = $4\frac{12}{17}$; then the square of AB is 100, and the square of BC is $22\frac{42}{289}$, and the difference of those squares is $77\frac{247}{289}$ the square of AC, and the square root of $77\frac{247}{289}$ is $8\frac{14}{17}$ the length of AC.

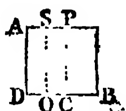
If AC = 4, and BC = 2, the sum of their squares is 20, and the square

root of 20 is the length of AB: but the square root of 20 is a surd: therefore AB and the other sides are *incommensurable.*

When AC and BC are equal, the hypothenuse AB is the diagonal of the same ACBD; and the square of AB is double the square of the side AC or CB: but twice a square number is not a rational square, or a square whose root can be exactly obtained; therefore the diagonal of a square and its side are *incommensurable*. In other words, whatever number of equal parts the side of a square may be divided into, the diagonal cannot contain an exact number of those parts.



84. *If a right line* (DB) *be divided into any number of parts* (DO, OC, CB), *the rectangles made by the whole line and each part, are together equal to the square on the whole line.*



Let AB be the square on the line DB; and from the points of division O, C, draw OS, CP, perpendicular to DB. Then because those lines are equal to DA or DB the side of the square, AO, SC, PB are the rectangles made by the whole line and each part respectively, and these rectangles together evidently constitute the square, because the whole is equal to all its parts taken together. Or if we denote the rectangles after the manner of products, AO is equal to DB $\times$ DO, SC equal to DB $\times$ OC, and PB equal to DB $\times$ CB, and the three products together equal to DB^2.

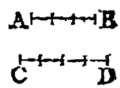
Of Ratios and Proportions which respect Magnitudes.

DEFINITIONS.

85. The following Definition of *Ratio* is usually given in the 5*th*. Book of Euclid's Elements.

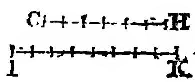
"Ratio is a mutual relation of two magnitudes of the same kind to one another in respect of quantity."

This definition is frequently objected to as imperfect and obscure. And it seems difficult to acquire a correct idea of the ratio of two magnitudes from the definition, if we are limited to the consideration of *magnitudes* abstractly. By the help of numbers however, it becomes perfectly intelligible.



Thus, if we divide the line or magnitude AB into 3 equal parts, and the magnitude CD contains 4 of those parts, the relation of AB to CD is the same as that of 3 to 4, which in numbers, is the ratio of the magnitudes AB and CD in respect of quantity.

Let GH be any other line or magnitude divided into 6 equal parts, and suppose IK contains 8 of those parts.

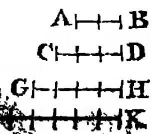


Then the relation or ratio of GH to IK is the same as that of AB to CD, because GH is contained or can be taken in IK as often as AB is contained or can be taken in CD, for the same reason that 6 is contained in 8 as often as 3 is contained in 4, that is, because $\frac{8}{6} = \frac{4}{3}$.

Those four lines or magnitudes are proportional; *viz.* AB is to CD, as GH is to IK; and are set down in the manner of proportional numbers, thus AB : CD :: GH : IK. And the proportion must evidently hold good whether AB and CD

are commensurable or incommensurable when compared with GH and IK.

86. Quantities of the same kind which are commensurable or can be divided into like parts, or parts of the same magnitude, may be compared in the same manner as we compare numbers in geometrical proportion (133, 134, *arith.*). Thus if AB contains 2; CD, 3; GH, 4; and IK, 6 equal parts, those lines, or magnitude will evidently have the same proportion as the number of equal parts into which they are respectively divided;



$$\begin{array}{rcccccccl} & AB & : & CD & :: & GH & : & IK, \\ & 2 & : & 3 & :: & 4 & : & 6. \\ \text{Or} & AB & : & GH & :: & CD & : & IK, \\ & 2 & : & 4 & :: & 3 & : & 6. \end{array}$$

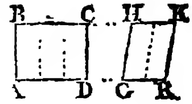
Or suppose the equal parts are again divided into a like number of equal parts, as 10 for example; then AB will contain 20; CD, 30; GH, 40; and IK, 60; therefore the quantities or lines will be in the proportion of 20, 30, 40, and 60; or as 2, 3, 4, and 6, the same as before.

Hence it is evident if we make use of a common measure, as in Practical Geometry that commensurable magnitudes may be represented by numbers, and their properties, as far as relates to proportion, demonstrated arithmetically. In the following theorems therefore, we shall sometimes refer to the articles in arithmetic which treat of proportion, in order to abridge the operations.

THEOREMS.

87. *Parallelograms* AC, GK *between the same parallels, or having the same altitude, are to one another in the same ratio as their bases* AD, GR.

For suppose AD is divided into 3 equal parts, and that GR contains 2 of those parts. Then if lines are drawn from the points of division parallel to the sides, the parallelogram AC will be divided into 3, and the parallelogram GK into 2 equal parallelograms, because they stand upon equal bases (82^a *corol.*)

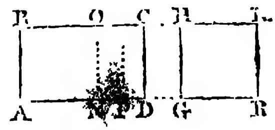


Therefore

3 is to [illegible] the paral. AC is to the paral. GK.

Or AD : [illegible] :: paral. C : paral. GK.

And if the bases AD, GR are incommensurable, the like proportion must evidently hold good.



Suppose the base GR is the side of a square, and the base AD its diagonal (83, *corol.* 1). Let AN = GR, and draw NO parallel to DC: and take NP so that AN and NP are commensurable.

Then, paral. BN : paral. BP :: AN : AP.

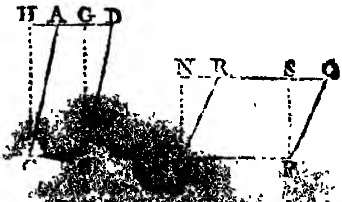
And by continually taking commensurable parts in the remainder PD, we may at last, approximate nearer to D than any assignable distance. Consequently the parallelogram BD will ultimately be to the parallelogram BN (or HR) as AD to GR.

Corol. 1. Since triangles are the halves of their parallelograms (82^a. *corol.* 1.), therefore triangles having the same, or equal altitudes, are to one another as their bases.

Corol. 2. If RK, and DC be taken for the equal bases of the parallelograms RH and DB, then RG and DA will be their altitudes: Therefore parallelograms, or triangles, on equal bases, are respectively in the same ratio as their heights.

88. *Parallelograms* CADB, ORQP, *having unequal bases and altitudes, are as the rectangles of the bases and altitudes.*

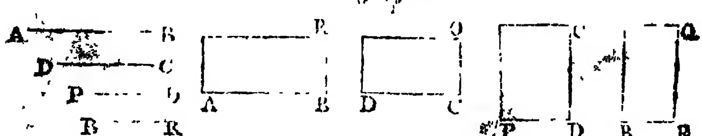
Make BG, CH, and PS, ON, perpendicular to CB, OP, respectively; then the rectangle HB is equal to the parallelogram AB, and the rectangle NP equal to the parallelogram RP (82). Then, because equals must have equal ratios, as rectangle to rectangle, so is parallelogram to parallelogram.



Scholium The parallelograms are also said to be in the compound ratio of their bases and altitudes. For if CB : OP, and BG : PS denote the ratio of the bases and altitudes, respectively, the rectangles of the corresponding terms or CB × BG : OP × PS will denote the compound ratio or the ratio of their rectangles. (41, [illegible])

Suppose CB = 2, BG = 5, OP = 4, PS = 3; then $\frac{2}{4}$ denotes the ratio of CB to OP; and $\frac{5}{3}$ that of BG to PS; and their product $\frac{2}{4} \times \frac{5}{3}$ (or $\frac{10}{12}$) is the compounded ratio or that of the parallelograms, namely, as 10 to 12.

89. *If four right lines* AB, DC, PD, BR *are proportional* (AB : DC :: PD : BR, *or* AB : PD :: DC : BR); *the rectangle* PC *made with the two means* DC, PD, *is equal to the rectangle* AR *made with the two extremes* AB, BR.



Let CO = BR, and RQ = DC. Then the rectangles AR, DO, having equal altitudes, will be as their bases (87); and for the same reason the rectangles PC, BQ will also be as their bases;

AB : DC :: *rectang*. AR : *rectang*. DO;
PD : BR :: *rectang*. PC : *rectang*. BQ;

But AB : DC :: PD : BR, therefore by equality of ratios *rectang*. AR : *rectang*. DO :: *rectang*. PC : *rectang*. BQ:

Now the surfaces or rectangles DO, BQ contained under the same or equal lines (DC, BR) must be equal; therefore the consequents being equal, the antecedents or rectangles AR, PC will also be equal.

Or thus: [illegible] two lines is analogous to the product of [illegible] AB : DC :: PD : BR, then AB × BR [illegible]0, *Arith.*)*

Corol. 1. When DC and PD are equal, the rectangle PC becomes a square; and its side is a mean proportional between the other two [illegible] AB and BR (151, *Arith.*).

Corol. [illegible] the product of the base and perpendicular g[illegible]e of a parallelogram.

---

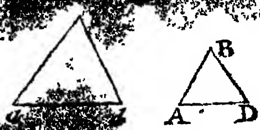
* Here the surfaces of the rectangles or parallelograms AR and PC are denoted by AB × BR, and DC × PD. And if AB = 8, BR = 3, DC = 6, and PD = 4 (inches for example), then 8 × 3 and 6 × [illegible] the surface or *areas* of those rectangles in square inches.

Of Similar Plane Figures.

Definitions.

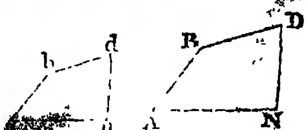
90. Similar rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportional.

Thus, if the angles of the triangles ABD, *abd* are respectively equal, and AB : BD :: *ab* : *bd*; and AB : AD :: *ab* : *ad*, &c. the triangles are said to be similar.



The sides opposite the equal angles are called homologous; thus AB, *ab* are homologous sides.

91. And if ABDN, *abdn* are equiangular, and AB : AN :: *ab* : *an* &c. the two figures are similar.

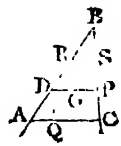


Corol. Hence all squares are similar.

92. All circles are similar.

Theorems.

93. *If one side of a triangle be divided into any number of equal parts, and from the points of division lines are drawn through the triangle parallel to one of the other sides, those lines will divide the third side into the same number of equal parts.*



Suppose BR, RD, DA are equal, and RS, DP parallel to AC. Then will BS, SP, PC, be equal to each other.

Draw RO, DQ, parallel to BC.

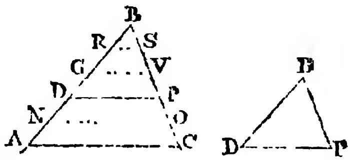
Then because the triangles RBS, DRO, ADQ are equiangular, and the like sides BR, RD, DA equal, those triangles will be identical or equal in all respects (86): consequently BS, RO, DQ, [illegible]

But RP, [illegible] parallelograms, therefore PC = DQ, and SP = RO, [illegible] each equal to BS.

Corol. Hence, if right lines AC, DP, RS, &c. cutting the sides of a triangle make the segments DA = DR, PC = PS, &c. those lines [illegible] parallel.

94. *If [illegible] in a triangle parallel to one of the sides, [illegible] the other two sides proportionally.*

Let DP be parallel to AC.



Then BD : BA :: BP : BC.
And BA : DA :: BC : PC.

Suppose BD is divided into 3 equal parts, and that DA contains two of those parts, and let lines be drawn from the points of division in BA parallel to AC, meeting BC. Then BC will also be divided into 5 equal parts (93).

Now, whatever part BD is of BA, the like part will PB be of BC, let the actual lengths of the equal parts in BA and BC be what they will: thus BD is $\frac{3}{5}$ of BA, and BP is $\frac{3}{5}$ of BC; therefore the relation or ratio of BD and BA is the same as that of BP and BC; consequently those four lines are proportional,

BD : BA :: BP : BC.

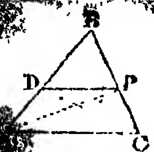
And because DA is $\frac{1}{5}$ of BA, and PC $\frac{1}{5}$ of BC, these are also proportionals,

BA : DA :: BC : PC.

And it is also evident that BD, DA, BP, PC are proportionals; for DA is the same part of BD, as PC is of BP (each being $\frac{1}{4}$);

Hence BD : DA :: BP : PC.

Otherwise th[illegible]



Join AP, DC. Then the triangles DAP, DCP, standing upon the same base DP, and between the parallels DP, AC are equal (82^a):

To each of these add the triangle [illegible] triangles BDC, BPA will be equal (32):

But the triangles BDP, BDC standing on the bases BP, BC and having the same vertex D, have the same altitude:

Also, the triangles BPD, BPA on the bases BD, BA, and having the vertex P, have the same altitude;

Therefore BD : BA :: triang. BPD : triang. BPA (87, *corol.* 1.)
And BP : BC :: triang. BDP : triang. BDC.

But the ratio of the triangle BPD to the triangle BPA is the same as that of BDP to BDC, because they are respectively equal:

Therefore the ratio of BD to BA is the same as that of BP to BC (31); or BD : BA :: BP : BC.

Corol. 1. Because the parallels AC, DP make the angles BAC, BDP equal, and the angle BCA = BPD (40), the triangles BAC, BDP are equiangular or similar: Therefore similar triangles have the sides about the equal angles proportional:

Thus the angle ABC of the triangle ABC, is equal to the angle DBP of the triangle DBP; and BD : BA :: BP : BC. And if the triangle BDP were applied to the triangle BAC so that the angles DPB, ACB are made to coincide, it may be proved in the same manner, that the including sides BP, DP, and BC, AC are proportionals.

And conversely, if [illegible] angles DB[illegible] of two triangles DBP, ABC [illegible] about those angles proportional, the [illegible] mutually equi-angular.

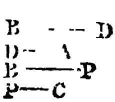
Corol. 2. Hence also, if lines (NO, DP, &c.) drawn through a triangle, are parallel to the base (AC), the intercepted segments of the sides (AN, CO; ND, OP, &c.) will be proportional:

[illegible] AN :: BC : CO;
[illegible] BC :: AN : CO (86 or 89):

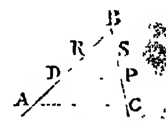
In like [illegible] : BO :: ND : OP:
But BA :BC :: BN : BO; hence AN.CO::ND:OP, by equality.

SCHOLIUM. Hence all that relates to the composition and division of ratios when these respect the comparison of right lines, will easily be comprehended: thus, *If 4 right lines are proportional,* BD : DA :: BP : PC, *they will also be proportional by composition and division.*

That is, BD+DA:BD(or DA)::BP+PC:BP(or PC).



And BD−DA:BD(or DA)::BP−PC:BP(or PC).



On two indefinite lines BA, BC meeting in B, take BD = BD, BP = BP, DA and DR each = DA, and PC, PS each = PC: then as the corresponding segments in BA and BC have the same ratio as

those sides, and the sides of the the triangles ABC, DBP, RBS are also proportional, we have

$$BA : BD :: BC : BP.$$

That is $$BD + DA : BD :: BP + PC : BP.$$

But DA and PC have the same ratio as BD and BP,

Therefore $$BD + DA : DA :: BP + PC : PC.$$

Again, $BD - DA =$ [illegible] BS, and the sides of the triangles BRS, RD[illegible]tional,

BR (or $BD - DA$) : BD :: BS (or [illegible]) : BP.

But BR and RD or DA, and BS and SP or PC are proportional,

Whence $$BD - DA : DA :: BP - PC : PC.$$

Also, because the sides of the triangles [illegible]RS are proportional,

$BD + DA : BD - DA ::$ BP + P[illegible]C.

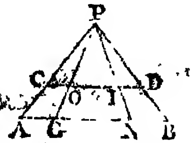
And if any number of right lines are proportional, BR : BS :: RD : SP :: DA : PC; then, *as any antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.* For BA is the sum of the antecedents, and BC that of the consequents, and the corresponding segments in BA, BC, in the same ratio as those sides, it will be

$$BR : BS :: BR + RD + DA \text{ (or } BA) : BS + SP + PC \text{ (or } BC).$$

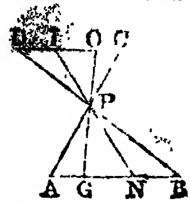
And the same will hold good with proportional quantities of any kind, for such magnitudes may be represented by lines, or by numbers. (Arith. *art.* 136).

95. *If several right lines meeting, or intersecting each other in a point P, are cut by two parallel lines AB, CD; the intercepted segments will be respectively proportional:*

AG : CO :: GN : OI :: NB : ID, &c.

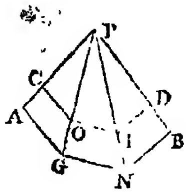


For the triangles APG, CPO; GPN, OPI; NPB, IPD are respectively equi-angular, and therefore similar.



Hence ([illegible]) AG : CO :: GP : OP :: GN : OI :: NP : IP :: NB : ID, &c.

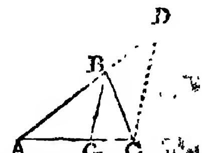
Therefore (31) AG : CO :: GN : OI : NB : ID, &c.



Corol. Hence it is evident, if AG, GN, &c. and CO, OI, &c. are not in the same continued right lines, but respectively parallel as before, that CO, OI, ID, &c. will be in the same proportion as AG, GN, NB, &c.

96. *The line BG bisecting the vertical angle ABC of the triangle ABC, divides the base AC into two parts having the ratio of the sides AB, BC.*

AB : AG :: BC : GC.



Draw CD parallel to BG meeting AB produced in D.

Then because CD is parallel to BG, the angles BCD, GBC are equal (40).

And the external angle CBA (or double the angle GBC) of the triangle CBD, is equal to both the angles BCD, BDC (41).

Therefore the angles BDC, BCD are equal, and consequently BD is equal to BC (46, *corol.* 2).

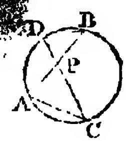
But the triangles ABG, ADC are similar;

Hence AB : AC :: BD (BC) : GC (94, *corol.* 1).

Corol. Hence, if a line bisects the vertical angle of a triangle, the rectangle of either side and the alternate segment of the base, is equal to the rectangle of the other side and the remaining segment:

$$AB \times GC = AG \times BC.$$

97. *In a Circle, if two chords* AB, CD *intersect each other, and their extremities are joined, the triangles* PCA, PBD *will be similar; and the rectangle of the segments* PA × PB *equal to the rectangle of the segments* PC × PD.

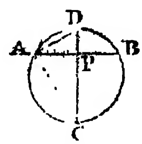


For the angles PBD, PCA, standing on the same arc DA, are equal to each other (70).

And the angles PDB, PAC, standing on the arc CB, are also equal.

And the equal angles at P being common to both triangles, those triangles are therefore equi-angular, and consequently similar;

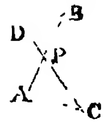
Hence PA : PC :: PD : PB (94, *corol.* 1):
Therefore PA × PB = PC × PD (89).



Corol. 1. If one chord DC bisects the other AB at right angles, then DC is the diameter of the circle (65), and AP or PB is a mean proportional between DP and PC.

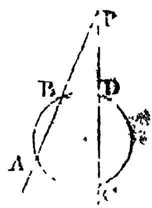
Corol. 2. And if AD, AC are joined, the angle CAD is a right one (72); therefore the perpendicular AP let fall from the right angle on the hypotenuse DC, is a mean proportional between the segments DP, CP.

Corol. 3. Hence also, if two lines AB, CD intersect each other in the point P, and PA × PB = PD × PC; then a circle will pass through the points D, B, C, A. And the triangles PDB, PAC, will be similar.



98. *If two right lines* PA, PC *from the same point* P, *intersect a circle, and the chords* BD, AC *are drawn; then the triangles* BPD, CPA *will be similar; and the rectangle* PA × PB *is equal to the rectangle* PC × PD.

For the sum of the two opposite angles BDC, BAC is equal to two right angles (71).



And the angles BDC, BDP are together equal to two right angles.

Therefore the angle BDP is equal to the angle BAC.

And for the like reason, the angle DBP is equal to the angle DCA.

And the angle P being common to both triangles, those triangles must be equi-angular, and consequently similar:

Hence PA : PC :: PD : PB,
Therefore PA × PB = PC × PD (80).

99. *If* PB *be a tangent to a circle, and* PC *a secant; then the rectangle* PC × PD *is equal to the square of the tangent* PB.

Draw BD, BC. Then the angle PBD is equal to the angle PCB (73).

And the angle ABC equal to the angle BDC (73).

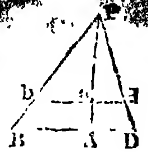
Therefore the angles ABC, BDC, being equal, their supplements or the angles CBP, BDP must be equal.

Consequently, the triangles PDB, PBC are equi-angular:

Hence PC : PB :: PB : PD.
Therefore PC × PD = PB².

100. *If two triangles* BPD, bPd *are similar; the bases* BD, bd, *and perpendiculars* PA, Pa, *are proportional:*
BD : bd :: PA : Pa.

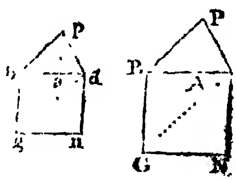
Because the angles BAP, *ba*P are right, the triangles BAP, *ba*P are also similar;



Hence PB : P*b* :: PA : P*a* (94, *corol.* 1),
And since PB : P*b* :: BD : *bd*,
Therefore BD : *bd* :: PA : P*a* (by equality).

101. *The surfaces or areas of similar triangles are in the duplicate ratio (or as the squares) of their homologous sides.*

Let the triangles BPD, *bpd* be similar; and BN, *bn*, the squares on the sides BD, *bd*.



Then triang. BPD : triang. *bpd* :: square BN : square *bn*.

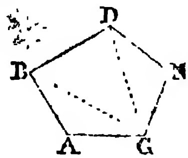
Suppose the perpendiculars PA, *pa*, are let fall on BD, *bd*, respectively; and join DG, *dg*.

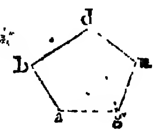
Then because the triangles BPD, BGD are on the same base BD, we have (87, *corol.* 2),
Triang. BPD : triang. BGD :: PA : BG (BD).
And, triang. *bpd* : triang. *bgd* :: *pa* : *bg* (*bd*).
But PA : BD :: *pa* : *bd* (100):

Therefore (31),

triang. BPD : triang. BGD :: triang. *bpd* : triang. *bgd*:
or triang. BPD : square BN :: triang. *bpd* : square *bn*;
because the two squares must evidently have the same ratio as their halves.

102. *All similar right lined plane figures* (ABDNG, abdng) *are to one another in the duplicate ratio, or, as the squares of their homologous sides* (AG, ag).





Draw GB, GD; *gb*, *gd*.

Then the figures being similar, the angle A is equal to the angle *a*; and the including sides AB, AG; *ab*, *ag*, are proportional (90); therefore the triangles ABG, *abg* are equi-angular and similar (94, *corol.* 1).

And if the equal angles ABG, *abg* are taken from the equal angles ABD, *abd*, the remaining angles GBD, *gbd*, must be equal.

Hence AB : *ab* :: BG : *bg*;

AB : *ab* :: BD : *bd*;

Therefore (31) BG : *bg* :: BD : *bd*: consequently (94, *corol.* 1) the triangles GBD, *gbd*, are similar. And in the same manner it may be proved that the triangles GDN, *gdn*, are similar.

Hence (101), triang. GAB : triang. *gab* :: GB² : *gb*² :: GBD : *gbd* :: GD² : *gd*² :: GDN : *gdn*; or GAB : *gab* :: GBD : *gbd* :: GDN : *gdn*.

And (94, *schol.*) GAB : *gab* :: GAB + GBD + GDN : *gab* + *gbd* + *gdn*.

But the antecedents together is the figure ABDNG, and the consequents the figure *abdng*;

Therefore AG² : *ag*² (GAB : *gab*) :: ABDNG : *abdng*.

To illustrate this by an example in numbers, suppose AG = 10 feet,

ag = 8 feet; and the area or surface of the figure ABDNG = 650 square feet;

Then $10^2 : 8^2 :: 650 : \frac{650 \times 64}{100} = 416$ square feet, the area or surface of $abdng$.

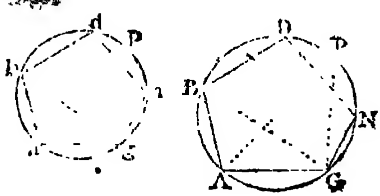
103. *The Perimeters of similar right lined plane figures are in the same ratio as their homologous sides.* (See the figures to the preceding Theorem.)

For the angles being equal, each to each, and the sides about the equal angles respectively proportional; we have

AG : ag :: GN : gn :: ND : nd :: DB : db :: BA : ba; therefore AG : ag :: sum of all the antecedents AG + GN + ND + DB + BA (the perimeter) : sum of all the consequents $ag + gn + nd + db + ba$ (the perimeter).

104. *The perimeters of similar Polygons* (ABDNG, abdng) *inscribed in circles, have the same ratio as the diameters* (AP, ap,) *of those circles.*

Draw GB, GP, gb, gp.



Then the polygons being similar, the triangles ABG, abg, will be equiangular, and the angle ABG equal to the angle abg (102).

But the angle APG is equal to the angle ABG; and the angle apg equal to the angle abg (70). And the angles AGP, agp, being right ones (72), the triangles APG, apg, are therefore equiangular.

Hence AP : ap :: AG : ag :: perim. of polyg. ABDNG : perim. of polyg. $abdng$ (103).

Corol. Hence it appears that the circumferences of circles have the same ratio as their diameters. For conceive regular

polygons of the like number of sides to be inscribed in both circles; then it follows that those polygons will be similar, and that their perimeters are in the same ratio as the diameters of the circles, let the number of sides be what they will. If now we suppose the number of sides to be continually augmented and their lengths diminished, it is manifest that at last, the differences between the perimeters and the circumferences of the circles, will be less than any assignable quantities; consequently the ultimate ratio of the perimeters and that of the circumferences must be equal.

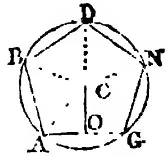
105. *The areas or surfaces of similar polygons inscribed in circles are in the duplicate ratio, or as the squares of the diameters of the circles.* (See the figures to the preceding Theorem).

For the triangles APG, apg, being similar, we have (101), $AP^2 : ap^2 :: \text{triang. } APG : \text{triang. } apg :: AG^2 : ag^2 :: \text{polyg. } ABDNG : \text{polyg. } abdng$ (102).

Corol. Hence, if we suppose (as in the last Theorem) the circumference of a circle to be the perimeter of a regular polygon, consisting of an infinite or rather an indefinite number of indefinitely short sides, it follows that the surfaces or areas of circles will be as the squares of their diameters. And because the circumferences are directly proportional to the diameters (104, *corol.*) the areas will be as the squares of the circumferences also.

106. *The area or surface of a polygon* (ABDNG) *is equal to a rectangle under half the perimeter and* (CO) *the distance of its centre from the sides.*

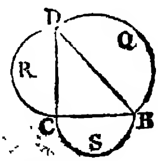
The centre of a regular polygon is a point equally distant from all its sides; and is the same as the centre of the inscribed, or circumscribing circle.



Suppose lines are drawn from the centre to the angular points; then the polygon will be divided into as many equal triangles as it has sides. And because those triangles are isosceles, CO will bisect AG and be perpendicular to it (46): therefore the area of the triangle ACG is half the rectangle CO × AG (89, *cor.* 2), or CO × $\frac{1}{2}$AG; and the area of another of the triangles (GCN) is CO × $\frac{1}{2}$GN, and so on; but the halves of all the sides together make half the perimeter; therefore the rectangle CO × *half the perimeter*, is the area of all the triangles or surface of the polygon.

Corol. Hence it appears, that the area or surface of a circle is equal to a rectangle under the radius and a right line equal to half the circumference. For, if we conceive the circle to be a regular polygon of an indefinite number of indefinitely short sides, the distance (CO) of the centre (C) from the sides, will in that case, be the radius of the circle, and half the perimeter becomes half the circumference.

107. *If semicircles* (Q, R, S,) *are described upon the sides of a right angled triangle* (BCD), *that which is upon the longest side* (DB) *will be equal to both the other two taken together.*



For circles being similar, and in the same ratio as the squares of their diameters (105, *corol.*) their halves must also be similar, and in like proportion, therefore

S : R :: CB² : CD², and by composition

S + R : R :: CB² + CD² (= BD², 83) : CD² :: Q : R,

or S + R : R :: Q : R; therefore S + R is equal to Q (31).

Hence, if similar figures are described on the sides of a right-angled triangle, that on the longest side will be equal to the other two taken together.

Of Planes and Solids.

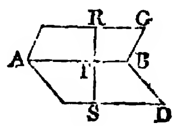
Definitions.

108. A right line is perpendicular to a plane when it is at right angles to all the right lines that can be drawn in that plane, from the point on which it insists.

109. The distance of a point from a Plane is a right line conceived to be drawn from that point perpendicular to the plane.

Corol. From the two preceding Definitions, and *Art.* 48, it follows, that a perpendicular is the shortest line which can be drawn from any point to the Plane.

110. The inclination of one plane to another is measured by the inclination of two right lines in those planes, drawn from any point in their common intersection, and at right angles to the same: Thus if AB is the line of intersection of the two parallelograms AG, AD; and PR, PS are perpendicular to AB, the inclination of the planes or parallelograms is the angle included by the lines PS, PR.

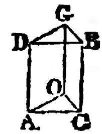


111. Parallel planes are those which are not inclined to each other, or are every where at an equal perpendicular distance.

112. A solid angle is that which is made by the meeting of more than two plane angles, which are not in the same plane, in one point.

113. Similar solid figures are such as have all their solid angles equal, each to each, and which are contained by the same number of similar planes.

114. A Prism is a solid whose ends are parallel, equal, and like plane figures, and its sides, connecting those ends, are parallelograms.



Thus AB is a triangular prism, its ends being the parallel and equal triangles AOC, DGB.

115. An upright prism is that which has the planes of the sides perpendicular to the ends or base.

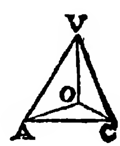
Thus AB is an upright prism; the sides, or parallelograms CG, GA, CD, being perpendicular to the ends or triangles AOC, DGB.

116. A Parallelopiped, or Parallelopipedon, is a prism bounded by six parallelograms, whereof the opposite ones are parallel, equal, and like to each other.

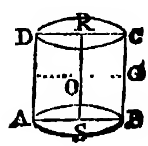
117. A rectangular parallelopipedon, or prism, is that whose bounding planes are all rectangles, and which stand at right angles one to another.

118. When all the bounding planes are squares, the prism or rectangular parallelopipedon, is called a Cube.

119. A Pyramid is a solid whose base is any right lined plane figure, and whose sides are triangles having all their vertices united in a point above the base, called the vertex of the pyramid.



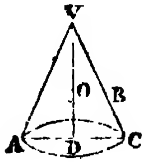
Thus AOCV is a triangular pyramid, its base being the triangle AOC, and its vertex V.



120. A Cylinder ABCD (sometimes called a round prism) is a solid conceived to be generated by the rotation of a rectangle SBCR about one of its sides SR, supposed at rest: which side SR is called the axis of the cylinder.

Corol. If OG is parallel to SB, those lines will describe equal circles; therefore every section of a cylinder parallel to its ends, is a circle equal to the base.

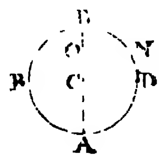
121. A Cone or round pyramid AVC is a solid generated by the rotation of a right angled triangle CDV about its perpendicular DV, called the axis of the cone.



Corol. If OB is parallel to DC, it will describe a circle; therefore the section of a cone parallel to the base is a circle.

122. Similar Cones, and Cylinders, are such as have their altitudes, and the diameters of their bases proportional.

123. A Sphere ARBD, is a solid supposed to be generated by the revolution of a semicircle (ABD) about the diameter (AB) which remains fixed, and is called the axis.



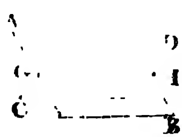
Corol. If ON is at right angles to the axis AB, it will describe a circle; therefore any section of a sphere, made by a plane, is a circle.

124. The altitude of a pyramid, or prism, is the perpendicular distance of the vertex, or upper plane thereof, from the plane of the base.

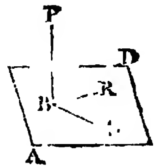
THEOREMS.

125. *The common section GH of two planes AB, CD is a right line.*

For let the extreme points G, H, of the common section be joined by the line GH, then that line being in the plane AB, and also in the plane CD (7.) it therefore must be the common section of both.

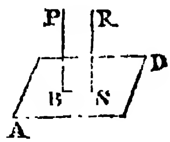


126. *If a right line PB be perpendicular to two right lines RB, SB, at their point of concourse B, it will be perpendicular to AD the plane of those lines.*



For suppose PB is perpendicular to a plane passing through the point B; then all right lines in that plane which meet in B will be at right angles to BP (108), therefore conversely, all right lines (RB, SB) which form right angles with BP at the point B, must fall in that plane.

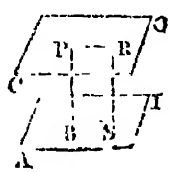
127. *If two right lines* (PB, RS) *are perpendicular to a plane* (AD,) *they will be parallel to each other.*



Join the points B, S. Then because BP is perpendicular to the plane AD, it must lie in (or is the common intersection of) every plane that passes through the point B which is perpendicular to the plane AD, it is therefore in the perpendicular plane that intersects AD in the line BS. In like manner SR must also lie in that same plane, or the perpendicular plane intersecting AD in the line SB; therefore as the angles PBS, RSB are right angles in the same plane, PB, RS, and are parallel to each other (40, corol. 2.).

Corol. Hence if several right lines are perpendicular to the same plane, they will be parallel to each other.

128. *If two planes* (AI, CO) *are parallel to each other, the right line* (PB) *which is perpendicular to one* (AI) *will also be perpendicular to the other* (CO).

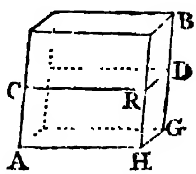


From any point S in the plane AI erect another perpendicular to that plane meeting the other plane in R, and draw PR, BS; then the planes being parallel, the two perpendiculars will be equal (111), and parallel (127); and as the angles at B and S are right angles,

whatever direction BS may be drawn upon the plane AI, the quadrilateral BPRS will always be a rectangle; consequently BP is perpendicular to PR or to the plane CO.

Corol. Since BS, PR are parallel, therefore the sections (BS, PR) made by a plane (BPRS) intersecting two parallel planes, are also parallel. And it is also manifest, that a plane will cut any number of parallel planes in like angles.

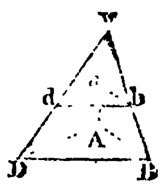
129. *If a Parallelopipedon or Prism* (AB) *be cut by a plane* (CD) *parallel to its base* (AG); *the section will be like and equal to the base.*



For by supposition the plane CD is parallel to the plane AG, therefore (128, *corol.*), the sections of those planes with the four sides of the prism are also parallel, namely, CR parallel to AH, RD parallel to HG, &c. and because the sides of the prism are parallelograms, the sides of the section CD will be equal to the corresponding sides of the base AG; therefore the section CD is a parallelogram like and equal to the base AG.

Corol. And the like is evident when the base is a polygon of any kind whatever: for the method of demonstration will be exactly the same if the sides of the prism are parallelograms.

130. *If a Pyramid* (DVAB) *be cut by a plane* (dba) *parallel to the base* (DBA), *the section* (dba) *will be similar to the base.*



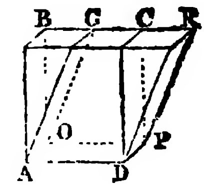
For (128, *corol.*) the sections *db*, *da*, *ab* are respectively parallel to DB, DA, AB, therefore the triangle *dVb* is similar to the triangle DVB, the triangle *dVa* to DVA, and *aVb* to AVB (91).

Hence DB : *db* :: BV : *b*V :: BA : *ba* :: AV : *a*V

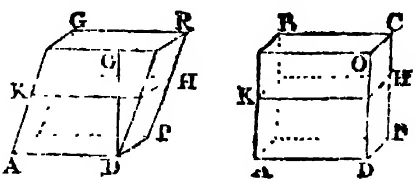
:: AD : ad; therefore db, ba, ad are as the corresponding sides of the base; and consequently the triangles dba, DBA are similar.

Corol. In like manner it is proved that all sections of a pyramid parallel to its base are similar, and similar to the base, whatever be the number of sides.

131. *Parallelopipeds or Prisms* (ABCD, AGRD) *on the same base* (AOPD), *and having equal altitudes, are equal to each other.*



By substituting surfaces for lines, and solids for surfaces, the demonstration will be similar to that in *Art.* 82, for parallelograms when BR is one right line. Thus, because the plane AB is parallel and equal to the plane DC, and the planes AG, DR also parallel and equal to each other, therefore BC is equal to GR; and taking GC, which is common to both those lines, from each, there remains BG equal to CR; consequently the solids ABGO, DCRP are bounded by like and equal planes, alike situated, and therefore are indentical: now if the solid ABGO is taken from the whole solid AR, the remainder is the prism AGRD; and the same whole AR lessened by the solid DCRP leaves the prism ABCD: therefore the two remainders or prisms AC, AR are equal (33).

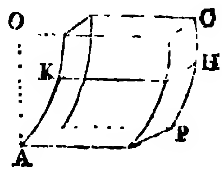


But the same conclusion is manifest from the *Method of Indivisibles*, which supposes that solids are composed of an indefinite number of indefinitely thin elementary parallel planes or sections: Thus, let AC, AR be the prisms having like bases AP, AP, and equal altitudes DO, DO; and conceive KH, KH to be two of those

indefinitely thin planes, parallel to the bases AP, AP: Then, as all the sections (KH, KH) are alike, and equal in both prisms, (129.) it is evident each prism is made up of exactly the same number of those equal elementary parts or sections lying one upon the other, those in AC vertically, and the others in AR, obliquely: which positions give their wholes or the two equal solids a different appearance.

The whole number of those indefinitely thin *laminæ* in each prism, is denoted by the perpendicular height DO; for if DO be divided into an indefinite number of parts, those parts, or the number of sections taken together, must again make up the whole line; hence it follows, that the base AP, or any section parallel to it, multiplied by the height DO, gives the sum of all the elements or the content of the prism.

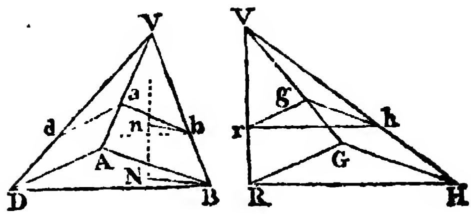
Corol. 1. Hence any solid AC having the base AP and height AO equal to those of the prism, will have the same magnitude as the prism, if all sections (KH, &c.) parallel to the base, are also equal to the base.



Corol. 2. And therefore it follows that prisms and cylinders of equal bases, and altitudes are also equal.

Corol. 3. Also because the base of a prism drawn into its height is the measure of its magnitude, therefore prisms are in the same proportion as their bases multiplied by the heights. Consequently if the bases are equal, the prisms will be as their heights; but in the ratio of their bases when the heights are equal.

132. *Pyramids* DVB, RVH, *standing upon the same, or upon equal bases* DAB, RGH, *and having equal altitudes* NV, RV, *are equal to each other.*



Let dab be a section parallel to the base DAB; and VnN the perpendicular from the vertex V upon the base DAB; and draw BN, bn: (the point n being in the plane dab).

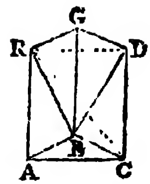
Then the triangles DVB, dVb are similar; and because Vnb, VNB, are right angles (126), the triangles VnB, VNB, will also be similar.

And since the triangles DAB, dab, are similar (130), their surfaces are as the squares of their homologous sides (101):

Hence, triang. DAB : triang. dab :: $DB^2 : db^2 :: BV^2 : bV^2 :: NV^2 : nV^2$: Therefore the sections DAB, dab, are as the squares of their distances from the vertex V. And in the same manner it is proved that the sections RGH, rgh are as RV^2 to rV^2. Now the bases DAB, RGH, and also the altitudes, being equal, the sections dab, rgh, at equal distances nV, rV, from the vertex, will also be equal. Therefore each pyramid is composed of a like series of indefinitely thin triangular sections (or *laminæ*); the greatest term of the series being the base DAB, or RGH, and the least o at the vertex V: consequently the two pyramids are equal. And when the bases are polygons of any kind whatever, the demonstration will evidently be similar to the foregoing.

Corol. Hence, if we suppose a circle to be a regular polygon of an indefinite number of indefinitely short sides (106, *corol.*), it follows that cones having equal bases and altitudes, are also equal. And that cones and pyramids of equal bases and heights are likewise equal to each other.

133. *A triangular pyramid is one-third of a prism having the same base and altitude.*



Let ABCDGR be a prism upon the triangular base ABC. Then if it be cut through the diagonal RC by the plane RBC; and through the two diagonals BR, BD, by the plane RBD, it will be divided into three equal pyramids ABCR, RGDB, and RDCB.

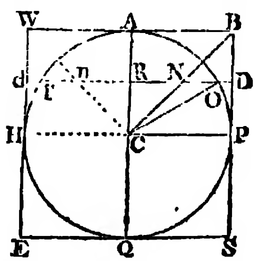
For if ABC is the base of the pyramid whose vertex is R, and RGD the base of the pyramid whose vertex is B, those pyramids and the prism will have equal bases and altitudes; therefore the two pyramids will be equal (132).

But the pyramids RDCB, ABCR, having the equal bases RAC, RDC, and the common vertex B, must also be equal, because in that case, their altitudes will be the same; therefore the three pyramids are equal to each other. And since the prism and (ABCR) one of the pyramids have the same base and altitude, the truth of the theorem is manifest *.

Corol. 1. Therefore prisms on polygonal bases are triple the pyramids on the same or equal bases, because the prisms may be divided into other prisms having triangular bases.

Corol. 2. And because prisms and cylinders, and pyramids and cones, having equal bases and altitudes, are respectively equal; therefore a cone is the third part of a cylinder of the same base and altitude.

131. *A sphere is two-thirds of its circumscribing cylinder.*



Let C be the centre of the circle circumscribed by the square WBSE; and draw CB.

Then if the rectangle QAES revolve about AQ as a fixed axis, the square CABP will describe the cylinder PHWB; the quadrant APC will

* A Learner will not readily comprehend this Theorem without models of the three pyramids.

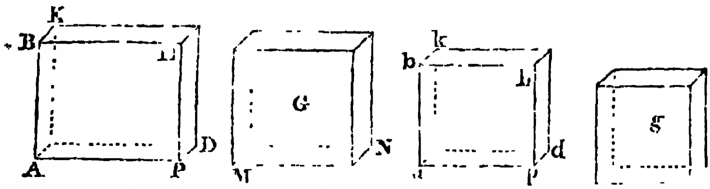
describe the hemisphere CPAH; and the triangle CBA will describe the cone CBW.

Let D*d* be parallel to PH, and join CO. Then the radius CO = CP = RD: and because AB = AC, RN will be = RC: but RO^2 $(RN^2) + RO^2 = CO^2$ (83) $= RD^2$; or $RN^2 + RO^2 = RD^2$. But semicircles described on RC (RN), and RO, are together equal to a semicircle described on CO (107.) or RD; therefore circles described on their doubles will also be equal, or the circle on D*d* equal to both the circles on N*n*, and OT: consequently N*n* the section of the cone, and OT the section of the sphere will together (in every section parallel to PH) be equal to D*d* the corresponding section of the cylinder. Now supposing the cylinder to be composed of indefinitely thin parallel sections (D*d*, &c.) then the cone on the same base (WB) being equal to one third of those sections, or ⅓ of the cylinder HB (133. *corol.* 2), therefore the hemisphere must be equal to the remaining ⅔ of that cylinder, or the whole sphere = ⅔ of the whole cylinder EB.

Corol. 1. A cone, hemisphere, and cylinder, of the same base and altitude, are in the proportion of ⅓, ⅔, and 1; or 1, 2, and 3.

Corol. 2. It also appears, that the spherical frustum HTOP, is equal to the difference between the cylinder H*d*DP and the cone C*n*N. And that the spherical segment TAO, is equal to the difference between the cylinder *d*WBD and the conic frustum *n*WBN.

135. *Similar upright prisms* BD, bd, *are in the same proportion as the cubes of their altitudes.*



Suppose G and g are cubes having heights respectively equal to AB and ab the heights of the prisms. Then prisms of equal altitudes being as their bases (131, *corol.* 3) we have

prism G : prism BD :: base MN : base AD,

or AB^3 : prism BD :: AB^2 : base AD, because the prism G is the cube of AB, and the base MN its square.

And in like manner, ab^3 (or prism g) : prism bd :: ab^2 : base ad.

But the parallelograms ABHP, $abhp$ are similar; and the bases AD, ad, are also similar; therefore (102),

$AB^2 : ab^2 :: ABHP : abhp :: AP^2 : ap^2 ::$ base AD : base ad;
or $AB^2 : ab^2 ::$ base AD : base ad,
or AB^2 : base AD :: ab^2 : base ad;

Whence by equality, AB^3 : prism BD :: ab^3 : prism bd, because the ratio AB^3 : prism BD, is equal to the ratio AB^2 : base AD, by the second of the above proportions, and the ratio ab^3 : prism bd equal to the ratio ab^2 : base ad, by the third.

If AK, ak, are made the bases, and AP, ap the perpendicular heights; then the prisms will be as the cubes of AP and ap: Hence,

Corol. 1. When four right lines AB, AP, ab, ap, are proportional, their squares, and also their cubes, will be proportional.

Corol. 2. And because similar plane figures are as the squares of their heights, or breadths, or other homologous lines in those figures, therefore similar prisms of any kind, and also cylinders, will be as the cubes of their like linear dimensions.

Corol. 3. Hence also, similar pyramids and cones, which are like parts of similar prisms and cylinders, will be in the same proportion as the cubes of their heights, or the diameters of their bases. And the like is to be understood of spheres, these being $\frac{2}{3}$ of similar cylinders.

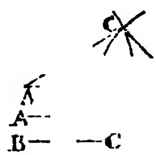
Scholium. This relation of similar solids is called Triplicate Ratio; and is sometimes demonstrated in parallelopipeds, by considering the ratio of the solids to be compounded of the ratios of the homologous linear dimensions. To give an exemplification in numbers: Suppose the bases AD, *ad*, are rectangular; and AB, AP, PD, are in the same proportion as 12, 15, 6; and *ab, ap, pd*, as 8, 10, 4; then the solid *bd*, will be to the solid BD as $8 \times 10 \times 4$ to $12 \times 15 \times 6$; therefore $\frac{8 \times 10 \times 4}{12 \times 15 \times 6}$ will denote the ratio of those products (92, *Arith.*): but this ratio is compounded of the ratios of the homologous sides, namely, of 8 to 12 or $\frac{8}{12}$, 10 to 15 or $\frac{10}{15}$, and 4 to 6 or $\frac{4}{6}$, and the compounded ratio is $\frac{8}{12} \times \frac{10}{15} \times \frac{4}{6}$ (141, *Arith.*) which, in its lowest terms is $\frac{8}{27}$, the ratio of the solids; but $\frac{8}{27}$ is the cube of $\frac{2}{3}$, the ratio of either two homologous sides.

PROBLEMS,

WITH THE

Method of Tracing the Figures on the Ground.

136. *To make a triangle with three given right lines* AB, AC, BC.



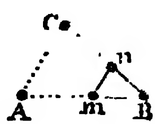
With the distances AC, BC as radii, about the centres A, and B, the extremities of the longest line, describe two arcs of circles intersecting each other in C; draw CA, CB. Then ABC is the triangle.

For the radii or two shortest sides of the triangle are, by construction, equal to the given lines AC, BC.

If both the shortest of the given lines together are less than the longest line, it is evident the arcs will not intersect each other, in which case the problem becomes impossible.

By means of this Problem, any right-lined figure may be copied: or a right-lined figure made exactly like another right-lined figure, first dividing the given figure into triangles.

A triangle may be marked on the ground by means of *cords*, or rather *measuring tapes* or *lines*: thus, suppose it is required to lay down the triangle ABC, whose sides shall be 60, 50, and 40 *feet*.



Having measured out AB = 60 *feet*, fasten the ends of two measuring lines at A and B; then draw them straight on the ground, and bring 50 *feet* on one line to 40 on the other, and where they intersect will give the point C.

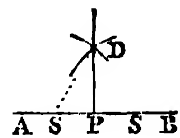
When the sides of the triangle are too long for the common measuring tapes or lines, lay down a triangle similar to that proposed, and then prolong the sides to the length required.

Thus suppose AB = 450, BC = 400, and AC = 300 *feet*. Take the same aliquot part of each side, $\frac{1}{10}$ for example (in the present case), or 45, 40, and 30 *feet*, and with those distances make the triangle B*mn*; then measure out BA = 450, and BC = 400; and if the triangle B*mn* is correctly laid down, AC will measure 300 *feet*. For, by similar triangles, 45 (B*m*) : 30 (*mn*) :: 450 (BA) : 300 (AC).

It is evident that any error in the length of *mn* will produce 10 times that error in AC; and therefore it may sometimes be necessary to repeat the operation more carefully.

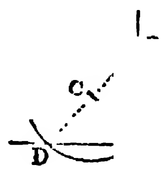
137. *At a given point* P *in a right line* AB, *to raise a perpendicular* PD, *to that line.*

On each side of P take equal distances PS, PS, and about S, S, as centres, with same radius, describe arcs intersecting each other in D; then draw PD for the perpendicular required.



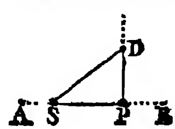
For if DS, DS, are joined, the triangle SDS will be isosceles; therefore (46, corol. 1), PD is perpendicular to SS or AB.

When the given point P *is near the end of the line.* About any convenient point C as a centre, describe a circle through P, cutting the given line in D, draw DCB, then join BP, which will be the perpendicular required.



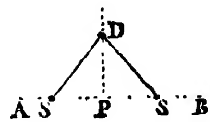
For DPB being a semicircle, the angle at P is a right one (72); therefore BP is perpendicular to DP.

This is readily performed on the ground by means of three *rods* or *lines*, whose lengths are in the proportion of 3, 4, and 5. Thus if the triangle SPD is laid down (by the preceding Problem) with SP = 16, PD = 12, and SD = 20 *feet*; then PD will be perpendicular to SP or AB (83)



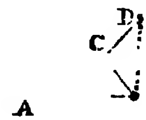
Otherwise thus:

Measure equal distances PS, PS on each side of P; then two rods or lines SD, SD, of an equal length, will make the triangle SDS isosceles; and consequently the direction of the perpendicular from P, is marked by the ends which meet at D.



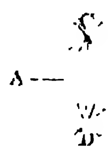
Or thus:

When the point P is near the end of the line. From any convenient point C make CS = CP, and CD = CS, S, C, and D being in a right line, then PD will be perpendicular to PA. For the angle DPS is a right one (7-).



138. *To bisect or divide into two equal parts, a given right line* AB.

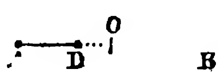
With any radius greater than half the given line, about the extremities A and B as centres, describe arcs intersecting each other in C and D: then draw CD, and it will bisect AB in the point P.



Draw the radii AC, AD, BD, BC: then those radii being equal, and the side CD common to both the triangles CAD, CBD, those triangles are therefore identical; and consequently the angle ACD is equal to the angle BCD. And since the triangle ACB is isosceles, AB is bisected by CP (46, *corol.* 1).

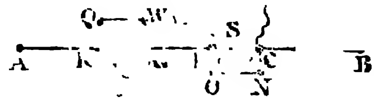
In this manner a line may be divided into 4, 8, 16, &c. equal parts. Thus AP, BP bisected give 4 equal parts; and those again bisected would make 8; and so on.

The most expeditious method of finding the middle of a line on the ground, is to measure equal distances from its extremities. Thus, suppose A and B are the ends of the line, and that AD, BC (found by measuring from A and B) are each 157 *feet*; and the remaining part DC is 19 *feet*; then O the middle of the line will evidently be $9\frac{1}{2}$ *feet* from D or C.



In measuring lines or distances on the ground, it sometimes may be necessary to take *off-sets* when obstacles fall in the way.

Suppose A and B are the extremities of a line to be measured: and that K and S are pools of water or swamps.



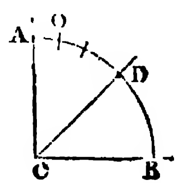
Having set up marks at R, G, D, C, in the line AB, measure equal *off-sets* CN, DO; and GW, RQ, at right angles to AB: then the quadrilaterals RW, DN being rectangular, QW will be equal to RG, and ON to DC; and the whole line AB equal to AR + QW + GD + ON + CB.

139. *To bisect a given right lined angle* ABC.

With any convenient radius BS, about the angular point B as a centre, describe an arc SR, and from the centres S, R, with any radius longer than half the distance between those points, describe two other arcs intersecting one another in D; then the line joining B and D will bisect the angle ABC, and the arc SR.

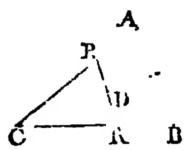
For if the radii SD, RD, are drawn, the sides of the triangles BRD, BSD will be respectively equal, each to each, therefore they are also equi-angular (46.), and consequently the angles RBD, SBD are equal.

By such bisections, an angle or its corresponding arc may be divided into 2, 4, 8, &c. equal parts. Thus if ACB be a quadrant, or an angle of 90 degrees (64.); the first bisection divides it into two equal angles, or the arc AB into two parts (DA, DB) of 45 *degrees* each: another bisection divides the arc AD into two equal parts of $22\frac{1}{2}$ *degrees*: the next gives an arc AO of $11\frac{1}{4}$ *degrees*: and if the bisection be continued 7 times, we get an arc of $42\frac{3}{16}$ *minutes*. Such a division is rea-



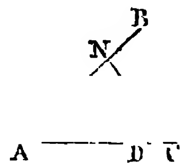
dily performed if the radius (CB) is 5 or 6 *inches*: and will be found convenient for measuring the degrees of an angle, when the usual instruments for that purpose are not at hand.

To bisect an angle ACB on the ground. Measure equal distances CR, CR, from the angular point C; then D, the middle of the cross distance RR, gives the direction of the line CD which bisects the angle. For the triangle RCR being isosceles, the line CD which bisects RR will also bisect the opposite angle (46).

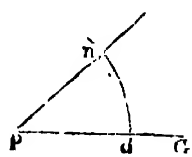


140. *At a given point* P *in a right line* PG *to make an angle* nPG *equal to a given right-lined angle* BAC.

About A and P with the same radius, describe arcs DN, *dn*; take *dn* equal to DN, and draw P*n*; then the angle *n*P*d* is equal to the angle NAD or BAC.



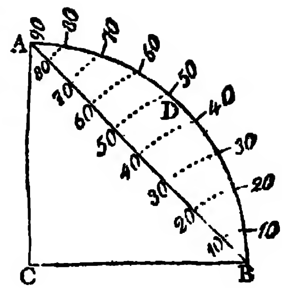
Draw the chords DN, *dn*. Then the corresponding sides of the triangles *d*P*n*, DAN being equal, the angles at P and A must therefore be equal (46[a]).



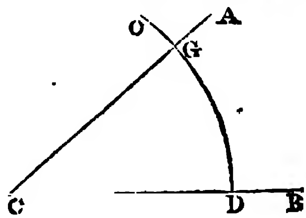
141. When it is proposed to make an angle which shall contain a given number of degrees, &c. (64); a *protractor, line of chords,* or a *sector*, will be necessary.

The common protractor is a semi-circular instrument for measuring and laying down angles. The arc or limb is divided into 180 equal parts or degrees; and when its centre is placed over the intersection of two lines, the number of degrees in the angle is shewn by the intercepted arc on the divided edge of the instrument. A protractor for the same purpose is frequently cut on the common plane scales, the centre being on one edge, and the graduations on the other.

142. A line of chords is made by transferring the divisions on the arc of a quadrant to its chord. Thus, suppose ACB is a quadrant, and the right line BA the chord of its arc BDA. Let this arc be divided into 90 equal parts or degrees: then if one foot of a pair of compasses be kept on the point B, and arcs successively described with the other, from each of the 90 divisions in BDA to meet BA, those arcs will divide it into a line of chords.



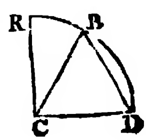
143. *To measure an angle with the line of chords.*—Suppose the angle ACB. With the radius CD equal to the extent of 60 degrees on the line, about the angular piont C as a centre, describe the arc DG; then the extent from D to G measured on the chords, gives the number of degrees, &c. contained in the angle: which, in this example, is about 40.



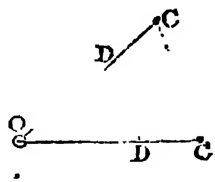
144. Hence the method of laying down an angle which shall contain a proposed number of degrees is obvious. Suppose for example, it is required to make the angle ACB of 40 *degrees*, CB being a given line. With CD the chord of 60 degrees, describe an arc DO as before; then 40 degrees taken on the line of chords, will extend from D to the point G in the arc through which the line CA must be drawn to form the required angle.

When the angles are greater than 90 degrees, measure, or lay them off at twice. Or produce one side so as to form two angles at the angular point, and then measure the supplement to 180 degrees.

The chord of 60 degrees is taken for the radius, because the sum of the angles of a triangle being 180 degrees (41), each angle of an equilateral triangle must therefore contain 60 degrees. Thus, if RCD is a quadrant, and the triangle BCD equilateral, BD (= the radius CD) is the chord of the arc DB, or of 60 degrees.

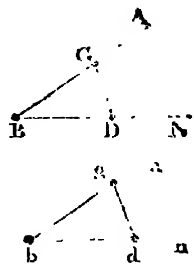


145. *To measure the angle* ACB *with the sector.* See the Fig. to *Art.* 143. About C with any radius CD, describe an intercepted arc DG. Open the sector till the distance between the brass points marked C, C, (the extremities of the chord-lines) is equal to the radius CD. Then if the distance DG be laid cross-ways on those chords, so that its extremities are equally distant from C, C, or from the centre of the instrument, the points of the compass will fall on the number of degrees in the angle. Thus if CO, CO, be the chord-lines of 60 degrees each on the sector, (moveable about the centre O) and DO the chord of any other arc, 40 degrees for example: then by similar triangles CO (the *radius*): DO (the *chord* of 40 degrees) :: CC : DD; therefore if CC be made the radius of any arc, or circle, DD will be the chord of 40 degrees in that arc, or circle.

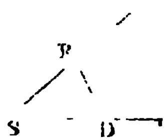


Hence it is, that the sector has frequently the advantage of the protractor, or common line of chords, because it may be set to different radii: the limits being the distance between the brass points C, C, when the instrument is shut, and their distance when it is quite open.

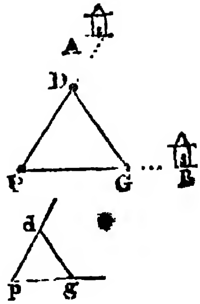
146. When it is proposed to trace an angle on the ground equal to another angle, the operation is similar to that in *Art.* 136. Thus, to lay down the angle *abn* equal to the angle ABN, the direction of *bn* being given. Measure equal distances BD, BG, and also the cross distance GD; then with those three distances lay down the triangle *bdg* (136), and the point *g* gives the direction of *ba*.



147. If the angle *abn* (when laid down) is to contain a given number of degrees; first, make an angle DSR on paper equal to those degrees; then having measured the equal sides SD, SR, and the opposite side RD on some convenient scale of equal parts, let the triangle *gbd* be traced on the ground with three corresponding distances in *feet* or *yards*, &c. (136). Thus, suppose the angle RSD is 41 degrees, then if SR, SD are each 40 on a scale of equal parts, RD will be 28 on the same scale, nearly: consequently if the triangle *gbd* is traced on the ground, with 40, 40, and 28 *feet*, the angle *abn* will be 41 degrees.

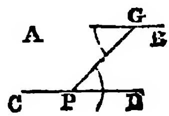


148. And therefore to determine nearly, the angle P subtended by two distant objects A and B, measure equal distances PD, PG, and the cross-distance DG; then construct a triangle *dpg* on paper, similar to DPG, and measure the angle *p* with a protractor, or the chords. Thus if PD, PG, are each 30 *feet*, and DG = $28\frac{1}{2}$ *feet*, the triangle *dpg* constructed with 30, 30, and $28\frac{1}{2}$ equal parts from any scale, will give the angle *p* (or P) = $56\frac{3}{4}$ *degrees*, nearly.

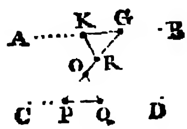


149. *Through a given point* P *to draw a line* CD *parallel to a given line* AB.

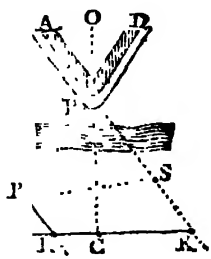
From P draw PG in any direction to meet the given line AB; then make the angle GPD equal to the angle AGP (140); and PD will be parallel to AB; because the alternate angles AGP, GPD are equal (40).



To trace the parallel CD *on the ground*; Fix on any convenient point G in AB, and measure an isosceles triangle RGK; then at the point P lay down the triangle OPQ equal to RGK; and PQ will be parallel to GK.



150. By means of this last problem we can bisect an inaccessible angle. Let it be required to determine the direction of the capital OP of a bastion. At any points B, S, in the directions of the faces DP, AP, set up two marks; and from B trace BR parallel to PS; measure equal distances BC, BR, and mark the point K in the direction CR; then find G the middle of CK: and the prolongation of GP will bisect the angle APD.

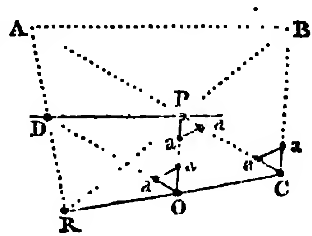


For the triangle CPK being similar to the isosceles triangle CBR, the line GP from the middle of the base CK bisects the opposite angle (16, *corol.* 1).

Corol. Hence if we measure CB, CR, CK, the distances CP, KP, are

found by similar triangles. For CR : CE :: CK : CP. And a perpendicular from B on CR will give the distance GP at another proportion.

151. When it is proposed to trace a line through a given point P parallel to an inaccessible line AB, set up marks at any convenient points C, R, in the directions AP, BP; next, by means of three equal isosceles triangles Caa, Paa, Oaa, trace PO parallel to CB, and OD parallel to PC; then the direction DP is parallel to AB.



For by construction OD is parallel to CA, and OP to CB; therefore the triangles ORD, CRA; and OPR, CBR, are respectively similar;

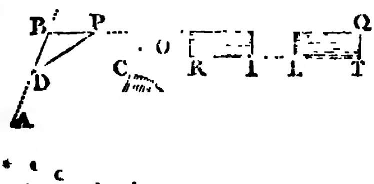
Hence RO : RC :: OD : CA,

And RO : RC :: OP : CB; therefore by equality of ratios, OD : OP :: CA : CB.

Now the sides about the equal angles DOP, ACB of the triangles DOP, ACB being proportional, those triangles are therefore similar (94, *corol.* 1); and since the homologous sides are respectively parallel and like situated, the third sides DP, AB must also be parallel.

Corol. Because the quadrilaterals RDPO, RABC are similar, if we measure the sides RO, DP, RC, the inaccessible distance AB may be found at one proportion; for RO : DP :: RC : AB.

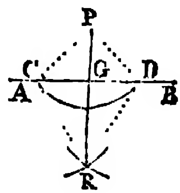
151[a]. In *castramentation* it is sometimes necessary to change the direction instead of continuing the fronts of all the battalions or divisions in the same line. Let QR be two divisions of the encampment, the fronts being in the same line OT, and IL the distance between them; and let it be required to place the other divisions GC, &c. that the fronts SG, &c. may be in a given direction or parallel to a given line BA, the distance between the divisions remaining as before or RS = IL, and (as is usual) the two prolongations RO, SO of the fronts equal to each other:



In BQ and BA take two equal distances BP, BD, and measure the sides of the isosceles triangle DBP; then the point O is found thus, DP : PB ::

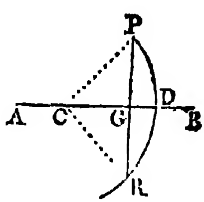
RS (or IL) : RO. Suppose BP = BD = 30, DP = 50, and RS = IL = 20 *feet*; then 50 : 30 :: 20 : 12 *feet* = RO. Therefore if RO be made = 12 *feet*, a string or tape OS = 12 *feet*, and another RS = 20, when stretched from O and R will give the point S, and the new direction OSG. For the triangles DBP, SOR being similar (94), and RO parallel to PB, the angles SOR, DBP are equal, and consequently OS is parallel to BD.

152. *From a given point* P *to let fall a perpendicular* PG *upon a given line* AB.



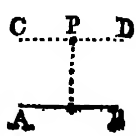
About P as a centre with any radius PD greater then the distance of P from AB, describe an arc DC; and from D and C with a radius greater than half DC, describe arcs intersecting each other in R; join PR: then PG is the perpendicular required.

Draw the radii RC, RD Then RC, CP being equal to RD, DP, respectively, and the side RP common to both the triangles RCP, RDP, those triangles are therefore identical, consequently the angles CPG, DPG are equal, and the triangle CPD isosceles; therefore PG is perpendicular to CD (46, *corol.* 1).

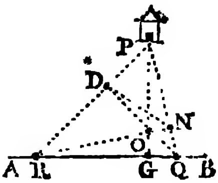


When the point is nearly opposite the end of the line. From any point C in AB, describe an arc PDR; take DR equal to DP; then join RP: and PG will be perpendicular to AB.

For by construction CD bisects the arc PDR in D; therefore PG is perpendicular to CD (65).



When a perpendicular is to be traced on the ground: First trace the line CPD parallel to AB (by 149); then a perpendicular to CD at the point P (137) will also be perpendicular to AB.



153. *If the object* P *is inaccessible:* Set up marks at any two convenient points R, Q, in AB; then on RP, QP, trace the perpendiculars QD, RN; and the point of intersection O gives the direction of the perpendicular POG.

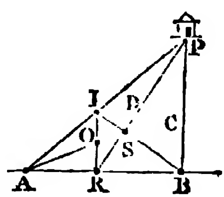
We have to prove that POG is perpendicular to AB.

Conceive DN to be joined: Then because the opposite angles ODP, ONP, of the quadrilateral ODPN are right angles, a circle will pass through the points O, D, P, N, (72), therefore the angles ODN, OPN, standing on the same chord ON (of the circle) will be equal to each other (70).

And since RDO, QNO are right angles, and the angle ROD equal to the angle QON, therefore the triangles RDO, QNO are equi-angular; hence DO : ON :: RO : OQ; therefore the triangles ODN, ORQ, are also equi-angular (94, *corol.* 1), consequently the angle ORQ = ODN = OPN. But the angles ORQ, GQN, together are equal to a right angle (41, *corol.* 2); therefore OPN and GQN make a right angle, and consequently PGQ is a right angle.

Corol. Hence the three perpendiculars let fall fro the angles of a triangle upon the opposite sides, will intersect one another in the same point.

Or thus:



Let AB be the line, and P th inaccessible object as before. At any convenient point R in AB, trace a perpendicular RO to AB, which continue till OI = RO. Make PIA a right line, then mark the point S where the lines AOS, and RP meet, also the point B or concourse of the lines AB and ISB. And PB will be perpendicular to AB.

For let IC parallel to AB meet AOC in C; then AR : RB :: CD : DI (95), and by composition (94, *schol*) AB : RB :: CI : DI. But because the triangles OIC, ORC are similar, and OI = OR, therefore CI = AR, hence the last proportion becomes

AB : RB :: AR : DI.

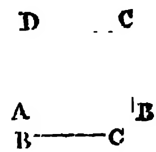
And AP : IP :: AR : DI, by the sim. triang. ARP, IDP;

Therefore AB : RB :: AP : IP (by equality); therefore RI, BP are parallel (94).

Corol. By this problem we may find the distance of an inaccessible object P from an accessible line AB. For if we measure AR and RB, it will be AR : RI .: AB : BP.

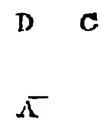
151. *On a given base* AB *to make a Rectangle whose height shall be equal to a given line* BC.

At the extremities of the base AB erect the perpendiculars AD, BC, each equal to BC; then join DC; and ADCB is the rectangle required.



For AB and DC being at the same perpendicular distance, they must therefore be parallel; and since the angles at A and B are right angles, the parallels AC, BC, will meet DC in right angles (40, *corol.* 2); therefore DB is a rectangle (22).

Corol. 1. In like manner a square is constructed on a given line AB by making the perpendiculars AD, BC, each equal to AB.



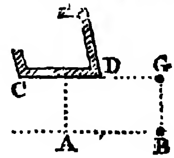
Corol. 2. Hence also, a line (DC) is drawn parallel to a given line (AB), at a given distance (BC).

155. The following is also a practical method of drawing a line DC parallel to another line AB, at a given distance PD.

With the given distance PD in the compasses, about any two points P, O, in AB, as centres, describe arcs D and C; then lay the edge of a ruler to touch those arcs, and draw the line required. For if PD, OC are drawn to the points of contact, PDCO will be a rectangle (67).

To trace a Rectangle on the Ground. Having measured out one side (the direction being given) to the required length, erect perpendiculars at its ends; then if those perpendiculars are prolonged to the distance proposed, their extremities will evidently mark the angular points of the Rectangle.

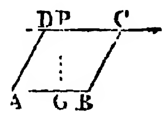
In like manner a line is traced parallel to another line inaccessible at one end, at a proposed distance from that other line. Let it be required to trace the line AB, parallel to the face of the bastion CD, at the distance of 300 yards.



Having taken the point G in the direction CD, make GB perpendicular to DG, and equal to 300 yards; then if BA is traced perpendicular to GB, it will be parallel to CD.

If a battery is constructed at A against the bastion, the shot (at right angles to AB) will strike its face CD in a perpendicular direction, or with the greatest force possible.

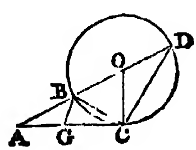
156. *On a given base* AB *to make a parallelogram* DB *of a given height* GP, *so that the sides* AD, AB *shall form a given angle.*



From any point G in AB erect the perpendicular GP equal to the given height (137); through P draw DC parallel to AB (149); and make the angle DAB equal to that proposed (140); then draw BC parallel to AD: and ADCB is the parallelogram. For the opposite angles being equal, the sides opposite those angles will also be respectively equal (80).

Corol. Hence from a given point (A) to draw a line (AD) to meet a given line (DC) in a given angle (BCP). At any point C in the given line, make an angle DCB equal to the angle proposed; then from the given point A draw AD parallel to CB; and the thing is done.

157. *To divide a given line* AC *according to mean and extreme proportion; or so, that the rectangle under the whole line and one part, shall be equal to the square on the other part: or* CA : CG :: CG : GA.



Make CO perpendicular to, and $= \frac{1}{2}$AC; about O as a centre with OC describe a circle; draw AOD, and join DC, and parallel to it draw BG. Then CA : CG :: CG : GA.

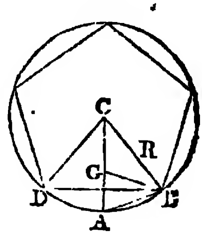
Join CB. Then the triangles ABC, ACD being similar (99) we have, AD : AC :: AC : AB, or AD : BD :: BD : AB (because BD = AC); therefore AD is divided in B according to mean and extreme proportion: And because BG is parallel to DC, it divides CA in the same proportion in G, as DA is divided in B (94, *corol.* 2).

Corol: Hence AB = GC. For because of the parallels BG, DC, we have AD : BD :: AC : GC (94).

And AD : BD :: BD : AB;

whence (by equality) AC : GC :: BD : AB; now the antecedents being equal, the consequents GC, AB, are necessarily so.

158. *In a given circle to inscribe a regular Pentagon.*



Having divided the radius CA (by the foregoing Problem) according to mean and extreme proportion in G, make GB = GC; take AD = AB; then draw BD, which will be the side of the pentagon, or the chord of $\frac{1}{5}$ of the circumference of the circle.

Draw GR parallel to AB: then CR = CG, and RB = GA (94).

By construction, CA : CG :: CG : GA, or because GB = GC, CB : GB :: GB : RB.

But the angle GBR is = the angle GCB, therefore the sides CB, GB; GB, RB about those equal angles, are proportional, hence the triangles BGC, BRG are equi-angular (94, *corol.* 1); therefore the former being isosceles, the latter BRG will also be isosceles, consequently RG = RB. But the outward angle GRC of the triangle GRB is equal to both the inward opposite angles, and therefore equal to twice the angle GBR; consequently the angle ABC, which is equal to GRC, is twice the angle GBR; therefore BG bisects the angle ABC. Hence in

the isosceles triangle ACB, each of the angles at A and B is double the other angle ACB.

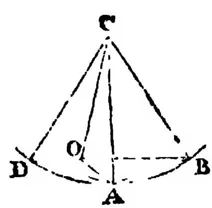
Now all the angles of the triangle ACB being $\frac{5}{5}$ of two right angles, the angle ACB is $\frac{1}{5}$ of two right angles, and its double, or the angle DCB $= \frac{1}{5}$ of 4 right angles: therefore DB is the chord of $\frac{1}{5}$ of the circumference: and 5 of those chords form the pentagon.

Corol. 1. Because the angles ABG, ACB are equal, and the angle CAB common to the triangles CAB, GAB, and the former isosceles, the latter GAB is also isosceles, and consequently AB = BG (= GC); therefore if the radius of a circle is divided according to mean and extreme proportion, the greater segment (GC = GB = AB) will be the side (AB) of a regular decagon in that circle.

Corol. 2. Hence also, BD bisects GA, and the angle GBA.

159. THEOREM. The square on the side DB of a regular pentagon inscribed in a circle, is equal to the square on the radius CB, and the square on DA the side of the decagon taken together (*Euclid, B.* 13. *Pr.* 10.):

Let CO bisect the angle DCA; and join OA.



The angle DCB is equal to $\frac{4}{10}$ } of 2 right
and DCO . . . . . . to $\frac{1}{10}$ } angles:
Therefore OCB is equal to $\frac{3}{10}$ of 2 right angles.

And each of the angles CDB, CBD is also equal to $\frac{3}{10}$ of 2 right angles:

Therefore the triangle COB is isosceles, and OC = OB:
Consequently the triangles COB, DCB are equi-angular:

Hence, $OB : BC :: BC : DB$, therefore the square on BC is equal to the rectangle under OB and DB (89, *corol.* 1).

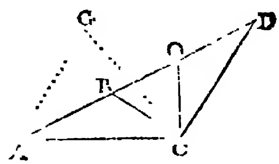
And because the triangles DOA, DAB are isosceles, and the angle ODA common, those triangles are similar:

Therefore $AD : DO :: DB : AD$; hence the square on AD is equal to the rectangle under DO and DB:

And therefore the sum of the squares on BC and AD is equal to the sum of the rectangles $OB \times DB$, and $DO \times DB$. But $OB \times DB + DO \times DB = DB^2$ (84), that is, the square on BC + the square on AD = the square on DB.

160. *On a given line* AC *to construct a regular Pentagon.*

Make CO perpendicular to and = $\frac{1}{2}$AC; through O draw AD to make OD = OC: join CD, and that will be the radius of the circle in which AC is the side of the Pentagon.



Take OB = OC. Then as the construction is analogous to that in *Art.* 157; AD will therefore be divided according to mean and extreme proportion in B; and consequently if AD is made the radius of a circle, BD will be the side of a Decagon in the same circle (158, *corol.*).

But the triangles ADC, ACB are similar (157):

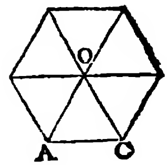
Hence $AD : DC :: AC\ (DB) : BC$;

or $AD : DB :: DC : BC$; therefore DC and BC are in the ratio of the radius of a circle to the side of the inscribed decagon. Hence, because BCD is a right angle, $CD^2 + CB^2 = BD^2$ (83); therefore if BD (AC) is the side of a pentagon, CB will be that of the decagon, and CD the radius of the circumscribing circle (159):

Therefore make AG, CG, each equal to CD; and G will be the centre of the circumscribing circle.

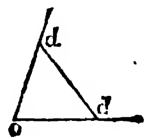
161. *On a given line* AC *to construct a regular Hexagon.*

Make AO, CO each equal to AC (136); and the triangle AOC will be equilateral and equi-angular; then 6 of those triangles, having each an angular point at O, will evidently form the required hexagon.



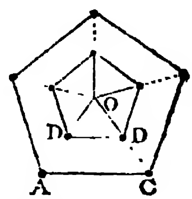
A Pentagon, or Hexagon, when the extent of the sides are too great for the common measuring tapes or lines, may be traced on the ground by means of proportional distances (136). Thus, suppose it is required to lay down a Pentagon whose side AC shall be 100 *yards*.

Having made the angle $dOd = 72$ degrees on paper (141), measure equal distances Od, Od on a scale of equal parts, suppose 80 each, then the distance dd will be 94 nearly on the same scale.



Lay down 5 triangles DOD, &c. with the equal sides OD, OD, &c. each equal 80, and DD, &c equal to 94 *feet* (136).

Then by similar triangles, 94 (DD) : 80 (OD) :: 300 (AC) : 255 *feet* nearly = OC: therefore if OA, OC, &c. are measured out to 255 *feet* each, their extremities will mark the angular points of the pentagon.



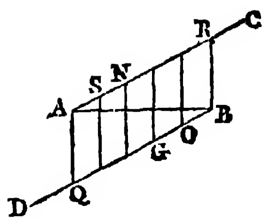
And the Hexagon may be traced on the ground in the same manner by means of 6 equilateral triangles.

But in tracing large and regular Works where exactness is required, the angles at the centres should be laid down with a Theodolite, and the distances to the angular points of the Polygons computed trigonometrically.

N. B. Of the common Geometrical Problems, the foregoing are among the most simple and necessary in Field-practice. It is easy to perceive however, that great accuracy cannot be expected, particularly when the Ground is not level.

162. *To divide a given line* AB *into a proposed number of equal parts: suppose 5.*

From the extremities draw AC, BD parallel to each other; in those lines take 5 equal parts of any convenient length (BO = OG &c. = AS = SN, &c.) join the opposite points of division; and AB will be divided into 5 equal parts.

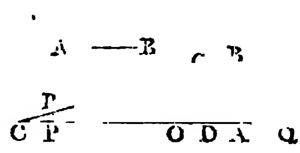


For BQ being parallel and equal to AR, AQ and BR will also be parallel and equal (80); therefore the lines joining the opposite points divide AB in the same proportion as the lines AR, BQ are divided (94).

Or thus:—Having drawn AC in any convenient direction, take the proposed number of equal parts AS, SN, &c. as before; then join RB, and parallel to it draw lines from the points of division in AR, and they will divide AB into the required number of equal parts (93).

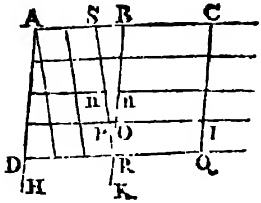
When the given line is too short to admit of distinct divisions, the following method is sometimes adopted to answer the same purpose.

Suppose AB is a given line to be divided into 7 equal parts. In a line CQ of any convenient length, take 7 equal parts, suppose from C to A; with CB = CA and AB = the given line AB, make the isosceles triangle CBA (136); take CP = DC, CR = CO, &c. and join the opposite points of division.



Then by similar triangles, CA : CD :: AB : DG; and because CD is $\frac{6}{7}$ of CA, DG will be $\frac{6}{7}$ of AB; and therefore OR = $\frac{5}{7}$, &c. and the shortest line PP = $\frac{1}{7}$.

163 Hence is derived the method of making *Diagonal Scales*. Let a Scale be constructed to *12ths*. of the line AB.



Having divided AB into 3 equal parts, draw two parallel lines AH, BK making any convenient angles with AB: in those lines take 4 equal distances, suppose from A to D, and from B to R; and through the points of division draw 4 lines parallel to AB; next, divide DR into 3 equal parts: then if the points of division in AB and DR are joined diagonally, the scale is constructed.

For by similar triangles, RB : BS :: RO : OP; therefore RO being $\frac{1}{4}$ of RB; OP will be $\frac{1}{4}$ of BS, or $\frac{1}{4}$ of $\frac{1}{3}$ (or $\frac{1}{12}$) of BA; and the next division *nn* is $\frac{2}{12}$, &c.

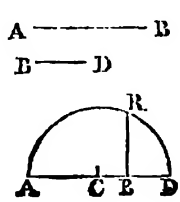
If QR = CB = BA is the scale for a *foot*, OP is an *inch*, *nn* = 2 *inches*, IP = 13 *inches*, &c.

But if we divide AB into 4 equal parts, only 3 must be taken in AH and BK to make 12*ths*. of AB (because 4 × 3 = 12).

Generally:—Resolve the number to which the divisions are to be extended, into two factors, then divide the given line (AB) into as many equal parts as there are units in one factor, and take as many equal parts in the other lines (AH, BK) as there are units in the other. Thus if AB is divided into 3 equal parts, and 5 are taken in AH, BK; or if AB is divided into 5, and 3 are taken in AH, BK, in either case the scale gives 15*ths* of AB. On the common Plain Scales, the equal parts in each line are 10, which give the divisions in 100*ths*.

A line divided into equal parts, and one of the parts subdivided, as in *Art* 162, or else diagonally, is called a *Line* or *Scale of Equal Parts*. A variety are to be found on the common Plain Scale belonging to a Case of Instruments.

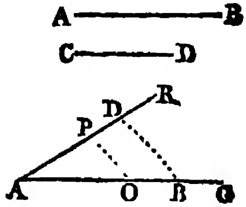
164. *To find a mean Proportional between two given lines* AB *and* BD.



Take AB and BD in one line AD, which bisect in C; and about C as a Centre, with CA or CD describe a semicircle; then if BR be drawn perpendicular to AD, it will be the mean proportional required (97, *corol.* 1).

165. *To find a third Proportional to two given lines* AB, CD.

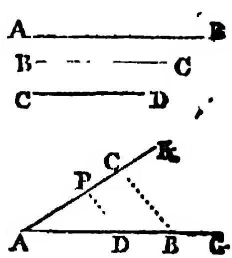
Draw two lines AG, AR making any convenient angle at A; take AB = AB, AD and AO each = CD; join BD, parallel to which draw OP: then AP is the third proportional required.



For OP being parallel to BD, the triangles ABD, AOP are similar; therefore AB : AD (AO or CD) :: AO : AP (94).

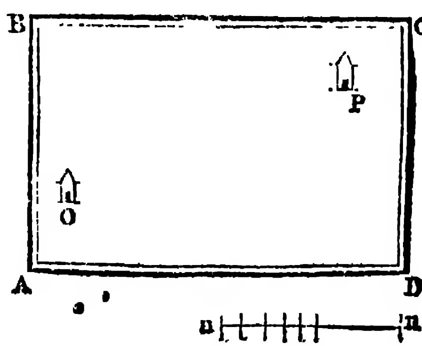
166. *To find a 4th Proportional to three given lines* AB, BC, CD.

Having taken two lines AG, AR, as in the foregoing Problem, make AB = AB, AC = BC, and join BC; then take AD = CD, and draw DP parallel to BC: By similar triangles AB : AC :: AD : AP the *4th.* proportional required (94).



167. This Problem is of very extensive use in the reduction of *Scales*, *Plans*, and *Maps*. We shall subjoin Examples.

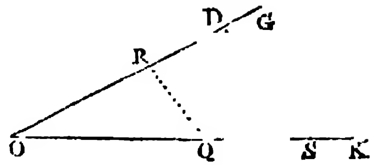
1. If ABCD be the Plan of a country, and suppose the distance between the objects O, P, is 1700 paces of a horse at $2\frac{3}{4}$ *feet* each, it is required to make a Scale of *yards* to the Plan.



$2\frac{3}{4} \times$ 700

yards.

Having drawn two indefinite lines OK, OG, forming any angle at O, make OS equal to the distance OP; and from any Scale of equal parts, set off OD = 1558, and OR = 1000; join DS, and parallel to it draw RQ, then OQ is a scale of 1000 *yards*. This divided, and subdivided is the Scale *mn*, in which each of the least divisions is 100 *yards*.

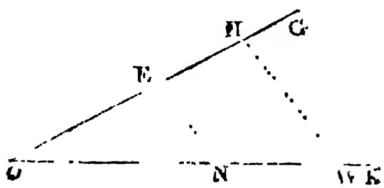


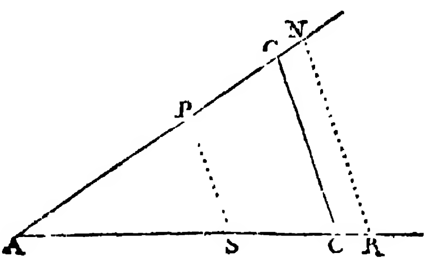
Or without the construction thus: The distance OP measured on a scale is 1·53 *inches*.

Then, as 1558 : 1·53 :: 1000 : 0·98 *of an inch*, the length of *mn* the Scale of 1000 *yards*.

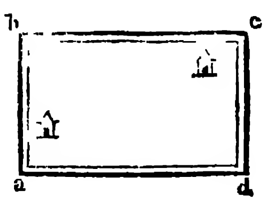
2. Let the Plan in the 1st Example be reduced to a Scale of 1 *inch to a mile?*

On two indefinite lines OK, OG (as in the last example), set off OE = 1000, and OH = 1760 (the yards in a mile) from any convenient scale of equal parts; and make ON = the scale *mn*; join EN, and parallel to it draw HW; then OW is the scale of a mile to the Plan ABCD.





Now with AC = OW, and CC = 1 *inch* (the two Scales) make the isosceles triangle ACC: then because any two corresponding distances on the Plans must be in the same proportion as the two scales, if AR be made equal to the length of the Plan ABCD, and AS = AB its breadth, RN and SP (which are parallel to CC) will be the length and breadth of the reduced Plan *abcd*.



Or, without the construction:

As 1558 : 1·53 *inch.* (OP) :: 1760 : 1·73 *inch.* length of a scale of 1 *mile* to the Plan ABCD.

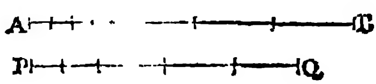
The length and breadth of the Plan ABCD are 1·89, and 1·15 *inches* respectively;
Hence, 1·73 : 1 :: 1·89 : 1·0[illegible] *inches*, the reduced length *ad*.
1·73 : 1 :: 1·15 : 0·67 .... breadth *ab*.

By means of the triangle ACC we may transfer any points or lines from one Plan to the other exactly in the same manner as the length and breadth *ad, ab,* were found. For any distances on ABCD being laid on AC, the proportional distances on the reduced Plan will be the corresponding parallels to CC. But the *Proportional Compasses* are peculiarly [illegible] for expedition in operations of this kind: Thus, shift the centre [illegible] instrument till, at the same opening, the extent of the points at one end is equal to one of the Scales (AC) and the extent of the points at the other end equal to the other Scale (CC). Then any opening or distance of the points at one end, will give the proportional or corresponding distance at the other. Or any two lines in the same proportion as the Scales may be used instead of the Scales themselves.

And *vice versâ*, any two corresponding distances on two similar Plans or Maps, and the length of one Scale, will give the length of the other.

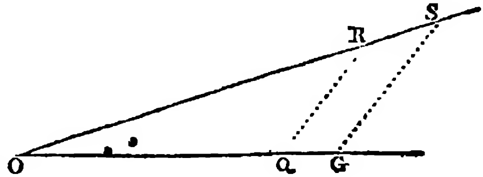
2. Suppose a Map is laid down to the scale AB of 4000 *Toises*; and that it be required to adapt a Scale (PQ) of *English miles* (4 for example) to the same Map.

The toise is = [illegible] yards [illegible], *Hutton*).



Therefore $\frac{2\cdot1315 \times 4000}{1760}$

= 4·84 *miles* nearly the Scale AB.



On two indefinite lines OS, OG, making any angle at O, set off OS = 4·84, and OR = 4 from any convenient scale of equal parts, make OG = the scale AB; join SG and draw RQ parallel thereto; then QO (PQ) is a scale of 4 miles.

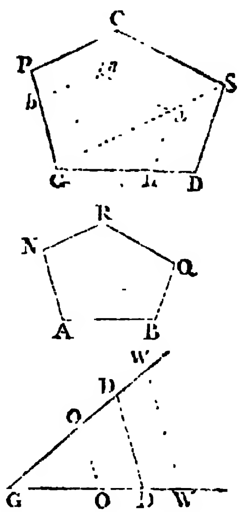
Or thus:—The length of the scale AB is 1·73 *inches*:

Therefore, as 4·84*m.* : 1·73*in* :: 4*m.* : 1·43*in* the length of the 4 mile scale PQ.

And the Map, or any part of it, may be enlarged, or diminished to a proposed Scale after the manner of *Examp.* 2. For we can suppose ABCD to be a given part of a large Plan.

4. On a given line AB to make a figure NB similar to a right lined figure PD.

With GD and AB make the isosceles triangle GDD; and draw the diagonals GS, GC: Then, as in the foregoing Examp. any lines of the figure PD being laid on GD, the corresponding lines of the required figure will be the parallels to DD. Thus if GW = the diagonal GS, and GO = DS; WW, and OO will be the diagonal AQ, and side BQ.



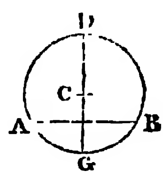
Or the figure may be constructed on the given one thus:
Make Gn = AB, then draw *na*, *aa*, and *ab* parallel to DS, SC, and CP.

An isosceles triangle is preferable to any other for these reductions, because the parallels (WW, OO) or 4*th.* proportionals are found with greater facility.

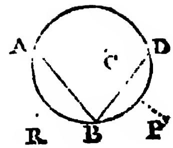
N. B. The foregoing constructions which respect the reduction of figures, are necessarily confined to a small scale; but the method may be extended to Plans, or Maps of any size whatever.

168. *To find the centre of a given Circle.*

Let any chord AB be bisected at right angles by GD, which therefore, will be a diameter to the circle: then C the centre of GD, will also be the centre of the circle (65, *corol.*).



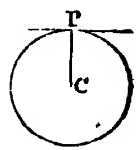
169. *Through three given points, not lying in a right line, to describe a circle.*



Let A, B, D, be the three points. Draw BA, BD, and bisect those lines with the perpendiculars RC, PC: then the intersection C is the centre of the circle (65, *corol.*), which described with the radius CA, CB, or CD, and the thing is done.

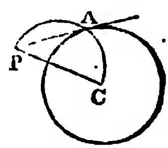
And in the same manner a circle is described about a triangle.

170. *Through a given point* P *to draw a Tangent to a circle.*

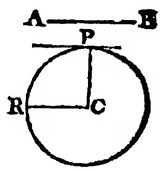


If P is in the circumference of the circle, draw the radius CP, then a line through P at right angles to PC is the tangent required (67, *corol.* 1).

When the given point P is without the circle: Draw PC to the centre C; and on PC describe a semicircle; then PA drawn to the intersection of the circles will be at right angles to the radius CA (72), and therefore a tangent to the circle.

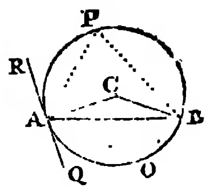


171. *To draw a Tangent to a circle parallel to a given line* AB.



Draw the radius CR parallel to AB. and make the radius CP perpendicular to CR; then a line through P, parallel to CR (and AB) will touch the circle in that point because it forms a right angle with the radius PC.

172. *On a given line* AB *to describe a Segment of a circle that shall contain a given Angle.*

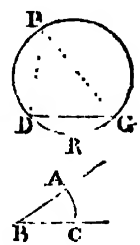


At the extremities A, B, of the given line, make each of the angles CAB, CBA equal to the difference of the proposed angle and a right one; and with CA or CB describe a circle: Then the segment APB on the same side of AB as the centre C, will contain the given angle when it is *less* than a right one; and the opposite segment AOB will contain it when it is *greater*.

For if RQ be a tangent at the point A, it will be perpendicular to the radius AC (67, *corol.* 1); then the angle CAB is the difference of the right angle CAQ and the angle BAQ; but BAQ is equal to the angle (APB) in the segment APB (73).

And the angle CAB is equal to the difference of the right angle RAC and the angle RAB, but this latter angle is equal to the angle (AOB) contained in the segment AOB (73).

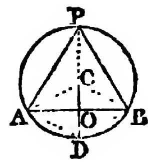
173. *To cut off a segment from a given circle that shall contain an angle equal to a given angle* ABC, *Or to draw a chord in a given circle that shall subtend a given angle at the circumference.*



About B with the radius of the circle, describe the arc AC; make the arc DRG = double the arc AC, and draw the chord DG: Then the angle (DPG in the segment DPG is equal to the angle ABC (69).

Or thus:—At any point (P) in the circumference, make an angle (DPG) equal to the given angle; then the line (DG) joining the extremities of the sides including the angle, is the chord required.

174. *To inscribe an equilateral triangle in a given circle.*

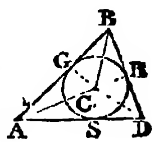


Bisect the radius CD at right angles with the chord AB; join BP, AP; and APB is the triangle.

Draw AD, BD, AC, BC. Then because OC = OD, and the side AO common, the triangles ACO, ADO will be identical (38); therefore AD is equal to the radius AC or CD; consequently both the triangles ACD, BCD, are equilateral; but the angle PAB = PDB, and PBA = PDA (70); and the remaining angle APB is = ACD or DCB (71); therefore the triangle APB is equi-angular and equilateral,

A *Square* is inscribed in a circle by joining the extremities of two diameters which intersect each other at right angles.

175. *To inscribe a circle in a given triangle* ABD.



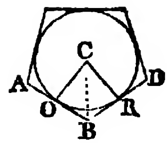
Bisect two of the angles, ABD, DAB, and from the intersection C of the bisecting lines, let fall perpendiculars CS, CG, CR on the sides; then if a circle be described about C with either of those perpendiculars, it will touch the sides of the triangle in S, G, and R.

For the two angles at A being equal, and the angles at G, S right ones, and the side AC common to both triangles AGC, ASC, those triangles are therefore identical (38); consequently the sides CG, CS are equal. And exactly in the same manner it is proved that CR and CG are also equal. Therefore the sides of the triangle will be tangents to the circle at G, R, and S (67).

Corol. Hence three lines bisecting the angles of a triangle, will intersect one another in the same point.

176. *To inscribe a circle in a regular Polygon* AD.

Bisect any two adjacent sides (BA, BD) with perpendiculars CO, CR; then their intersection C is the centre of the inscribed circle.

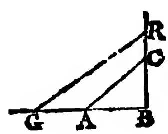


Draw BC. Then the hypotenuse BC being common to both the right angled triangles BOC, BRC, and BO = BR, the squares on OC, RC, will therefore be equal to each other (83, *corol.*), and consequently OC = RC. In like manner it is proved that the perpendiculars bisecting the other sides are all equal and meet in the same point C. Therefore a circle described with CO or CR will touch all the sides of the polygon (67).

And it is also evident that CB is the radius of the *circumscribing circle;* but this line bisects the angle ABD: Therefore to circumscribe a regular Polygon with a circle; bisect an two of its angles (except opposite ones) and the intersection of the bisecting lines is the centre of the circle.

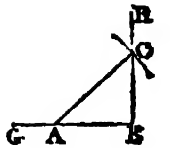
177. *To make a square equal to two given squares.*

Let BA, BC, the sides of the given squares be drawn to form a right angle ABC; join AC, which will be the side of the square required (83).



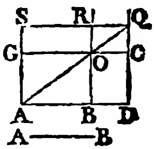
And in the same manner a square may be made equal to three, or more squares. For example, suppose the sides of three given squares are AB, BC, and BG; then because the square on AC is equal to the squares on AB, BC, if BR be made equal to AC, it follows that a square on GR will be equal to the three proposed squares.

178. *To make a square equal to the difference of two given squares.*



With two indefinite right lines BG, BR, make a right angle B; take BA equal to the side of the less square; and about A as a centre with AC the side of the greater. describe an arc to intersect BR in C; then BC is the side of the required square (83, *corol.*).

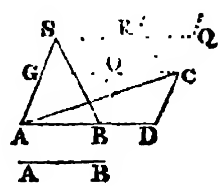
179. *On a given line* AB *to make a rectangle equal to a given rectangle* AGCD.



On AD (produced if necessary) take AB = the line AB: draw BR parallel to DC; and through O let AQ be drawn to meet DC produced; then if QS is made parallel to DA, BRSA will be the rectangle required.

For the triangles ASQ, ADQ; and ORQ, OCQ being respectively equal (80), the quadrilaterals ASRO, AOCD must therefore be equal (33); but the former, together with the triangle AOB, and the latter with the triangle AOG, make the two rectangles BRSA, AGCD, those rectangles must therefore be equal to each other, because the triangles AOB, AOG are equal.

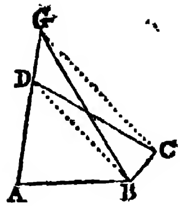
180. *On a given line* AB *to make a triangle* ASB *equal to a given triangle* ADC.



Draw AG and CG parallel to DC, and DA, respectively; then the parallelogram GD will be double the given triangle ADC (82^{a}, *corol.* 1): take AB equal to the given line AB; and by the construction in the preceding Problem, make the parallelogram AR equal to the parallelogram GD; draw the diagonal SB; and the triangle ASB will be equal to the given triangle ADC.

For the parallelograms AGCD, BRSA being equal, their halves must also be equal.

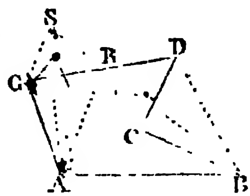
181. *To make a Triangle equal to a given Quadrilateral* ABCD.



Parallel to the diagonal BD draw CG to meet AD produced; join BG: then the triangle ABG is equal to the given quadrilateral.

For the triangles BCD, BGD on the same base BD, and between the same parallels BD, CG, are equal (82[a]), therefore the triangle ABD, together with the triangle DBG is equal to the same triangle ABD together with the triangle BCD (32), or the triangle ABC equal to the quadrilateral ABCD.

182. *To make a Triangle equal to the irregular pentangular figure* ABCDG *on the side* AB.

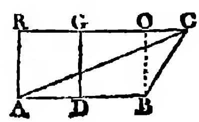


Let CR be drawn parallel to BD, and join BR. Then the triangles CBR, CDR, on the side CR and between the parallels CR, BD are equal (82[a]); therefore the figure ABCRG with the triangle CBR, is equal to the same figure together with the triangle CDR, and consequently the given figure ABCDG is reduced to the quadrilateral ABRG.

Now the quadrilateral ABRG is reduced to a triangle by the preceding Problem, thus:—Parallel to the diagonal AR draw GS to meet BR produced; then join AS; and the triangle ASB will be equal to the quadrilateral ABRG, and therefore equal to the given figure ABCDG.

And in like manner any multi-lateral right lined figure may be reduced to a triangle.

183. *To make a rectangle equal to a given triangle* ABC.



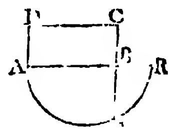
Let the base AB be bisected in D; and draw CR parallel to AB; then if AR, DG are made perpendicular to AB, the rectangle RD will be equal to the triangle ABC.

Draw BO parallel to DG. Then the triangle ABC is equal to half the rectangle RB (82, *corol.* 1): but RD is half the rectangle, therefore it is equal to the triangle ABC.

And therefore a rectangle whose height is half AR, and base AB will also be equal to the triangle.

184. *To make a square equal to a given rectangle* ABCD.

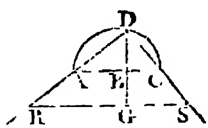
Extend AB till BR = BC, and on AR describe a semicircle, then produce CB to G; and the square on BG will be equal to the rectangle under AB, BR or BC (164).



Schol. Hence by this, and the preceding Problem, a square may be made equal to a given triangle: and consequently equal to any given right-lined figure (182).

185. *To make a rectangle of a given magnitude having its sides in the ratio of two given right lines.*

Let AB and BC be the given lines. Upon their sum AC describe a semicircle, and make BD perpendicular to AC; produce DB (if necessary) till DG is the side of a square equal to the given magnitude; join DA, DC, and through G draw RS parallel to AC meeting DA, DC produced (when necessary): Then RG, GS are the sides of the required rectangle.

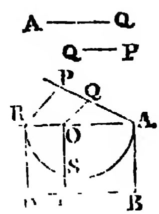


For the angle RDS being a right one (72), and DG perpendicular to RS, therefore the rectangle RS × GS is equal to the square on DG (97, *corol.* 2), or equal to the given magnitude (by the construction); and because RS is parallel to AC, the sides RG, GS, are in the given ratio of AB to BC (95).

186. *If* AC *is a square on the line* AO, *and* AQ, QP *two given right lines; to find another square that*

shall be to the square AC, *as* AQ *is to* QP. *Or, to find two squares having the ratio of two given right lines.*

In any convenient direction from A, take the given lines AQ, QP; join QO, and parallel there to draw PR to meet AO produced (if necessary): then if a semicircle be described on AR, OS will be the side of the required square.



Complete the rectangle RC. Then because QO, PR are parallel, the triangles AQO, APR will be similar,

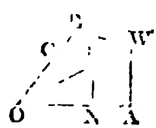
Hence, AQ : QP :: AO : OR (94, *corol* 2). But the parallelograms or rectangles AC, OD having the same height (AB or RD) will be in the ratio of their bases AO, OR (87):

Therefore AQ : PQ :: AO^2 (*rectang.* AC) : *rectang.* OD (= OR × OC or OA):

But the rectangle OR × OA is equal to OS^2 (97, *corol.* 2): and consequently AQ : QP :: AO^2 : OS^2, the required square.

187. *To describe a figure* (CRNO) *similar to a given right-lined figure* BWAO, *so that the latter may be to the former, as the line* AQ *is to the line* QP.

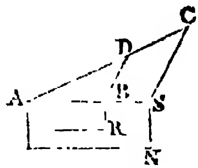
Find, by the last Problem, a square (OS^2) so that AQ : QP :: OA^2 : OS^2; and make ON = OS; draw NR, RC parallel to AW, WB, respectively; and CN is the figure required.



For the figures CRNO, BWAO being similar (165, *Ex.* 4); and because similar plane figures are as the squares of their homologous sides (102), we have BA : CN :: AO^2 : ON^2 (OS^2) :: AQ : QP (by the construction.)

By this Problem, plane figures are augmented, or reduced in Area according to any given proportion.

188. *To make a triangle* (ACS) *of a given magnitude, which shall also be similar to a given triangle* ADB.



On AB make the rectangle AR = to the given triangle ADB (183); then on AB (produced if necessary) let the rectangle AN be constructed equal to the magnitude of the required triangle, having its sides AS, SN in the ratio of AB to BR (18 , draw CS parallel to BD, meeting AD produced : and ACS is the triangle.

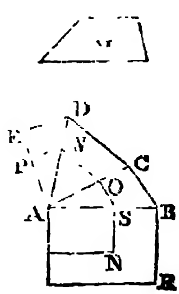
For the triangles ADB, ACS being similar, and also the rectangles AR, AN, we have (102),

rectang. AR : *rectang.* AN :: $AB^2 : AS^2$ :: *triang.* ADB : *triang.* ACS :

Or, *rectang.* AR : *rectang.* AN :: ADB : ACS; and the antecedents being equal (by the construction) the consequents AN, ACS must also be equal, or the triangle ACS = the given magnitude (by construction.)

Schol. Therefore a triangle may be made similar to one triangle and equal to another.

189. *To describe a figure* (ASOWP) *similar to a given figure* ABCDE, *and equal to a given right-lined figure* G.

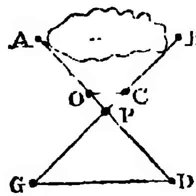


Let the two figures EB, and G be reduced to squares (181, 184.). Then the construction will evidently be exactly the same as that of the preceding problem. For if the rectangle AR be made equal to the figure EB, and a similar rectangle AN equal to G (183), the side AS of that rectangle will be the base of the required figure: then the sides SO, OW, WP being drawn parallel to the corres-

ponding sides of EB, the figure PS will be similar to EB, and equal to AN or G.

Methods of determining distances by means of similar Triangles traced on the Ground.

190. *To find the length of the line* AB *accessible only at both ends.*



Having fixed on some convenient point P, measure BP and AP; and prolong those lines till PG = PB, and PD = PA; then the distance between the points D and G will be equal to AB.

For the sides of the triangles GPD, BPA about the equal angles at P are respectively equal, therefore the third sides GD, BA will also be equal (38).

Or thus,

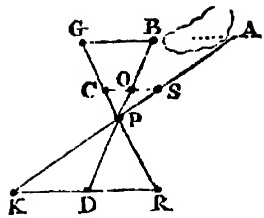
Having measured PB, PA (as before), take PC some convenient aliquot part of PB, and PO the same aliquot part of PA; then measure the cross distance OC, which will be the like aliquot part of the required distance AB.

For the sides PO, PA, PC, PB being proportional, the triangles OPC, APB will be similar;

Hence PO . PA :: OC . AB, therefore whatever part PO is of PA, the like OC will be of AB (91).

Suppose PA = 392, and PB = 411¼ *feet*; and let PO, PC be $\frac{1}{5}$ of PA PB, or equal to 78 *f.* 5 *in.* and 82 *f.* 3 *in.* And suppose OC measures 93½ *feet*; then AB = 93½ × 5 = 467½ *feet.*

191. *When the line* (AB) *is accessible at one end* (B) *only.*



We suppose some object at the inaccessible end A: and let a mark be set up at B: then in the direction AB take BG (the longer the better), and through a convenient point P, as in the preceding problem, let the distances BD, GR be measured, so that PD = PB, and PR = PG; then if a mark be set up at K the intersection of AP and RD when produced, DK will be equal to AB.

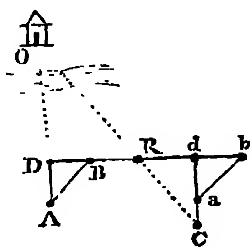
For the triangles PBG, PDR being similar and equal in all respects, the triangles PBA, PDK will also be similar and equal (95).

Or BA may be found without the distances PD, PR, thus; take PC, PO, like aliquot parts of PG, PB; then SO will be the same aliquot part of BA (95).

For PO : PB :: OS : BA.

Suppose PB = 442, PG = 464 *feet*; and that PO = 110½, PC = 116 *feet* (¼ of PB and PG), also, suppose OS measures 113 *feet*; then BA = 452 *feet*: for 110½ : 442 :: 113 : 452.

192. *Let* O *be an object on the opposite side of a river; to find the distance* DO.



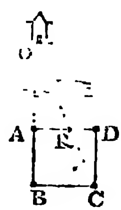
Lay down an isosceles triangle DBA, the side DB being in any convenient direction; then having measured a base DR, set up a mark at R; and in the same direction take another base R*d*, and make the triangle *dba* similar and equal to DBA (*da* being parallel and equal to DA): then find the concourse (C) of the lines ORC, *da*C, and measure *d*C:

By similar triangles, R*d* : *d*C :: RD : DO.

Suppose DR $=$ 300, R*d* $=$ 80, and *a*C $= 270\frac{1}{3}$ *feet,*

Then 80 : $270\frac{1}{3}$ :: 300 : 1014 *feet nearly* $=$ DO.

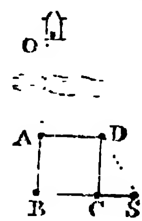
193. But the most expeditious method of finding the distance to an inaccessible object, is by means of a Rhombus, as follows:



Suppose O *the object, and* OB *the required distance.*—With a line or measuring tape whose length is equal to the side of the intended rhombus, lay down one side BA in the direction BO, and let BC another side be in any convenient direction: fasten two ends of two of those lines at C and A, then the other ends (at D) being kept together, and the lines stretched on the ground, those lines AD, CD will form the other two sides of the rhombus. Set up a mark at R where CO, AD, intersect; and measure RD:

Then the sides of the triangles RDC, CBO being respectively parallel, the triangles will be similar; hence, RD : DC :: CB : BO.

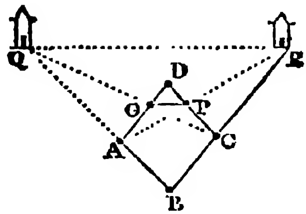
Suppose the side of the rhombus is 100 *feet,* and RD $=$ 11*f.* 7*in.* then $11\frac{7}{12}$: 100 :: 100 : 863 *feet nearly* $=$ BO.



Or thus:—Having laid down the rhombus, mark the concourse of the lines ODS, BCS, and measure CS: Then CS : CD :: AD : AO.

If the ground is nearly level, a rhombus whose side is 100 *feet* will determine distances to the extent of 300 yards within a very few feet of the truth.

194. *To find the length of an inaccessible line* (QR) *by means of a rhombus.*



At some convenient point B, lay down the rhombus (BADC), so that two of its sides BA, BC are directed to the extremities of the line. Mark the intersections O and P (as in the first case of the preceding problem): then the triangle ODP will be similar to the triangle RBQ; and OP parallel to QR.

For each of the rectangles DO × BQ, DP × BR being equal to the square on the side of the rhombus (as in the preceding prob.) they must therefore be equal to each other, or DO × BQ = DP × BR; therefore DO : DP :: BR : BQ; and since the angles at D and B are equal, the triangles ODP, RBQ will be similar (94, *corol.* 1). Therefore OD : OP :: RB : RQ.

Suppose OD = 9*f.* 5*in.* PD = 11*f.* 10*in.* OP = 13*f.* 7*in.* and the side of the rhombus = 100 *feet.*

Then $11\frac{10}{12} : 100 :: 100 : \frac{10000}{11\frac{10}{12}} = \text{RB}.$

Therefore $9\frac{5}{12}$ (OD) : $13\frac{7}{12}$ (OP) :: $\frac{10000}{11\frac{10}{12}}$ (RB) : $\frac{10000 \times 13\frac{7}{12}}{9\frac{5}{12} \times 11\frac{10}{12}} =$ 1219 *feet* = RQ.

Therefore the inaccessible distance RQ is found by multiplying the square of the side of the rhombus by OP, and dividing that product by the product of OD and PD.

The length of an inaccessible line may also be found by tracing a quadrilateral, as in *Art.* 151. Both methods however, are necessarily confined to moderate distances, and require much care in the execution in order to bring out satisfactory results.

PLANE TRIGONOMETRY.

DEFINITIONS.

195. A TRIANGLE has three sides and three angles: And any three of those being given (the three angles excepted) the others are found by means of similar triangles: This is the business of TRIGONOMETRY.

196. Hence it follows that Plane Trigonometry will admit of four different Cases: For the *data* may be

1. One side and two angles.

   (Therefore the 3*d*. angle is also given, *Art*. 41).

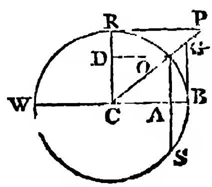
2. Two sides and an angle opposite to one of them.

3. Two sides and their included angle.

4. The three sides.

197. The sides of the similar triangles (or lines proportional to those sides) which enter into the computations, are called *sines*, *tangents*, *secants*, &c.

198. Let C be the centre of a circle, CR a radius perpendicular to the diameter WB, and PCB an angle, its measure being the arc OB (64, 144).



Draw the chord OS and BG perpendicular to the radius CB; and OD, RP perpendicular to the radius CR.

Then,

$$\left.\begin{array}{l}\text{AO is the Sine}\\ \text{AC or OD the Cosine}\\ \text{BG the Tangent}\\ \text{RP the Cotangent}\\ \text{CG the Secant}\\ \text{CP the Cosecant}\end{array}\right\}\begin{array}{l}\text{of the arc OB,}\\ \text{or angle PCB.}\end{array}$$

199. The *Cosine, Cotangent*, &c are the *Sine, Tangent*, &c. of the *complement* of the angle PCB to 90 *degrees*, or a right angle (*co* being a contraction of *complement*).

Thus, OD or AC is the *Sine*,
RP the *Tangent*,
CP the *Secant* of the angle PCR which is the complement of the angle PCB to a right angle; for the angles PCB, PCR together make the right angle BCR.

200. The *Sine, Tangent*, and *Secant* of an angle PCB are also the *Sine, Tangent*, and *Secant* of its *supplement* PCW, or the difference of PCB and 180 *degrees*.

201. AB is the *versed sine* of the arc OB or angle PCB: and AW the *versed sine* of the angle PCW.

202. When the arc is a quadrant or 90 *degrees*, its *sine* is the *radius*, and *cosine* 0: But the *tangent* and *secant* are infinite, because they become parallel and therefore do not meet.

Thus, CR is the *sine* of 90 *degrees* or the right angle RCB.

203. The degrees, minutes, &c. contained in an arc or angle are usually marked thus, °, ′, ″, &c. So 29°, 57′, 42″ denote 29 *degrees*, 57 *minutes*, 42 *seconds*.

204. *Corollaries.*

1. Hence it appears that (AO) the *sine* of an arc (OB) is half the chord (OS) of twice that arc (65).

2. Because the lines in and about the quadrant RCB form equiangular triangles, we have,

CA : AO :: CB : BG,
or, *cosine* : *sine* :: *radius* : *tangent*.

And, CA : CO :: CB : CG. Therefore the *radius* is a mean proportional between the *cosine* and *secant* of an angle.

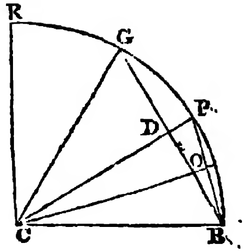
And BG : BC :: CR : RP. Hence the *radius* is also a mean proportional between the *tangent* and *cotangent*.

3. When the angle is 45° or half a right angle, the *sine* and *cosine* are equal; and the *tangent* and *cotangent* each equal to the *radius*.

And other properties are easily derived from the same figure.

Of computing the Sines, Cosines, &c.

205. SINCE 4 right angles contain 360° × 60 or 21600 *minutes*, it follows that $\frac{1}{2}$ the side of a regular polygon of 10800 sides inscribed in a circle, is the *sine* of an arc or angle of 1′. Half the side of a polygon of 5400 sides, is the *sine* of 2′. And half the side of a polygon of 2700 sides, the sine of 4′, &c. But these figures cannot be inscribed geometrically; for which reason the formation of the *Trigonometrical Canon*, or Tables of *Sines, Tangents*, &c. has been attended with much labour. Before fluxions were invented, the method of approximation was by continual bisections, which brought out chords corresponding to arcs in a descending geometrical progression; in this manner, the chord of a small arc being obtained, the chords of other small arcs were inferred from analogy on a supposition that the chords and arcs are nearly proportional when the angles are small: To explain this,



206. Let CRB be a quadrant. Make the chord BG equal to the radius CB; then the triangle CGB being equilateral, the angle GCB or arc GPB will contain 60°. Draw CP to bisect the chord BG; then GD or BD is the sine of 30° or the angle GCP or BCP. And if CO be drawn to bisect the chord BP, OP will be the *sine* of 15° the angle PCO, &c.

If the radius CB or CG is 1, then GD is = 0·5 the *sine* of 30°: and the *cosine* CD is equal to the square root of the difference between the squares of CG and GD (83, *corol.*).

The square of 1 is 1, and the square of 0·5 is 0·25, their difference is 0·75, whose square root is 0·86602540378 &c. = CD the *cosine* of 30° or *sine* of 60°, which taken from the radius CP (1) and the remainder is 0·13397459621 &c. the *versed sine* DP.

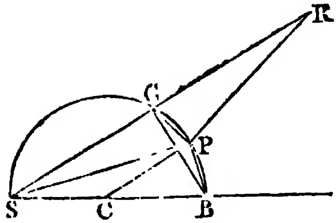
Now the chord PB is equal to the square root of the sum of the squares of DP and DB (83), and is found to be 0·51763809 &c. its half is 0·258819045 &c. = PO the *sine* of 15°.

And the *cosine* CO is = 0·965925826 &c the square root of the difference of the squares of CP and PO.

And the next bisection would give the *sine* of $7\frac{1}{2}$ degrees. But this method, though perhaps the most obvious, must evidently be extremely tedious. The like bisections, however, may be obtained with much greater facility by means of the following

207. THEOREM. *If any arc* BPG *of a semi-circle be bisected in* P; *then the chord* SP *is a mean proportional between the radius* CP, *and the chord* SG *and diameter* SB *taken together.*

Produce SG till GR is equal to the diameter SB, and join PR; then SR is equal to SG and SB.



Now the quadrilateral SGPB being in a circle, the external angle PGR is equal to the angle PBS (75). And because BS = GR, and PB = PG, therefore in the triangles PGR, PBS, the sides about the equal angles PGR, PBS are equal, therefore the triangles are identical, and consequently the third sides PR, PS are also equal. And because the angles PSC, PSG are equal (70, *corol.*), the isosceles triangles SPR, SCP will therefore be equiangular.

Hence CP : SP :: SP : SR, or $SP^2 = CP \times SR$.

Now if the radius CP be 1, SP^2 will be equal to SR, and SP equal to the square root of SR, or equal to the square root of the sum $2 + SG$, (because $GR = SB = 2$).

Hence, if the supplemental chord SG, of any arc (BG) be increased by the diameter (2), the square root of the sum will be the supplemental chord (SP) of half the arc (BG).

208. Let the chord BG be equal to the radius, then BPG is an arc of 60°. And because the angle SGB is a right one (72), SG is equal to the square root of the difference of the squares of SB and BG (85, *corol.*).

The square of SB is 4, and the square of BG is 1, therefore SG the supplemental chord of the arc BG or $\frac{1}{6}$ of the circumference is 1·73205080756887 &c. the square root of 3.

Consequently SR is = 2 + 1·73205080756887 &c. and its square root is 1·93185165257813 &c. SP the supplemental chord of the arc BP or $\frac{1}{12}$ of the circumference.

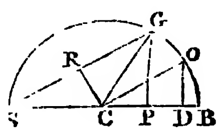
And the square root of $2 + 1 \cdot 93185165257813$ &c. is $= 1 \cdot 98288972274762$ &c. the supplemental chord of $\frac{1}{2}$ the arc BP, or $\frac{1}{24}$ of the circumference.

In this manner, after eleven bisections, we get $2 + 1 \cdot 99999973854478$ the square of the supplemental chord of $\frac{1}{12288}$ of the circumference or $1' \, 45'' \frac{15}{32}$: Which taken from 4, the square of the diameter, leaves $0 \cdot 00000026145522$ the square of the chord of $1' \, 45'' \frac{15}{32}$: And the square root is $0 \cdot 00051132692$ the chord of $1' \, 45'' \frac{15}{32}$, or the side of the inscribed polygon of 12288 sides *.

Now the chords of small arcs being nearly in the same proportion as the arcs themselves, we have, $1' \, 45'' \frac{15}{32} : 0 \cdot 00051132692 :: 2' : 0 \cdot 0005817764$ the chord of the arc of $2'$; and its half or $0 \cdot 0002908882$ is the sine of $1'$.

And the *cosine* is $= 0 \cdot 9999999577$ the square root of the difference of the squares of the radius 1, and the *sine*.

209. The *sine* and *cosine* of $1'$ being given, the *sine* of $2'$ *will be equal to twice the product of that sine and cosine.*



For let B be the centre of a circle, and OD, DC the *sine* and *cosine* of the arc OB or angle OCB, and GP the sine of GB or twice the arc BO. Then if CR be perpendicular to SG it will also bisect it (65). And because the angles OCB, GSP are equal (71), and CO equal to SC, the triangles SRC, CDO will be equal, therefore SG is equal to twice the *cosine* CD, and the triangles SPG, CDO are similar:

$$\text{Whence, } CO : OD :: SG \; (2CD) : GP;$$

Therefore when the radius CO is $= 1$, GP is $= 2CD \times OD$ (89).

$$\text{Again } CO : CD :: SG \; (2CD) : SP:$$

---

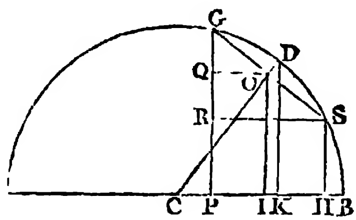
* *See* Ludolph Van Ceulen *de Circulo et Adscriptis*, where the bisections are continued 30 times, and the supplemental chords brought out to 40 places of figures.

And if CO is 1, SP is $= 2CD \times CD$, or $2CD^2$; and the *cosine* CP equal to the difference of SP and SC, or the difference of $2CD^2$ and 1.

Therefore if the angle OCB is 1′, GP the *sine* of 2′ will be $= 2 \times 0{\cdot}9999999577 \times 0{\cdot}0002908882 = 0{\cdot}0005817764$.

And the difference of twice the square of 0·9999999577 and 1, is 0·9999998308, the *cosine* of 2.

210. But to find the *sine* of 3, or the *sine* of triple an arc when its *sine* and *cosine* are given,



Let SH be the *sine* of the arc BS; OS or OG the *sine* of half the arc SDG; and GP the *sine* of the sum of both arcs BS and SG. Then DS is the common difference of the arcs BS and BD, and BD and BG; and therefore the arc BD is an arithmetical mean between the arcs BS, BG.

On CB let fall the perpendiculars DK, OI, and draw SR, OQ parallel to BC: Then because OS, OG are equal; QG and QR will also be equal (93): and RG being the difference of the *sines* HS, PG, therefore QG or QR is half their difference, and OI half their sum.

Now the triangles COI, CDK being similar, we have CD : DK :: CO : OI; therefore when the radius CD is 1, OI will be $= CO \times DK$:

But DK is the *sine* of the mean arc BD: therefore the product of DK the *sine* of the mean arc, and CO the *cosine* of the common difference DS, is equal to OI: consequently $2CO \times DK$ is equal to twice OI or the sum of HS and PG: and therefore PG is equal to the difference of $2CO \times DK$ and HS.

Now if the arcs BS, SD, DG are each 1′, then HS is the *sine* of 1′; CO is the *cosine* of 1′; and DK is the *sine* of the arc BD of 2′:

Therefore to find PG the *sine* of 3′, multiply twice the *cosine* of 1′ by the *sine* of 2′, and subtract the *sine* of 1′ from the product.

And if the arc BS is 2′, and SD, DG each 1′; then DK is the *sine* of 3′, and PG that of 4′:

And PG is equal to twice the *cosine* of 1′ × *sine* of 3′ *minus* the *sine* of 2′.

In like manner, if BS = 3′; SD and DG each = 1′; PG the *sine* of 5′ will be = twice the *cosine* of 1′ × *sine* of 4′ *minus* the *sine* of 3′: and so on for the *sine* of any multiple of the arc 1′.

Corol. If the mean arc BD is 60°, then CK the *sine* of 30° will be equal to $\frac{1}{2}$ CD (205); and because the angle IOS is the complement of IOC to a right angle, the triangle RGS is similar to the triangle OCI or DCK, therefore RG will be = $\frac{1}{2}$ GS (the arc BD being 60°) or the *sine* of the arc DG or DS; consequently PG (or PR + RG) will be equal to SH + OG:

Therefore if two arcs be taken, one greater than 60°, and the other as much less, the *sine* of the greater arc will be equal to the *sine* of the less arc, together with the sine of the arc which is the common difference from 60°.

Thus if the two arcs are 15° and 45°; then 0·2588 &c. the *sine* of 15° added to 0·7071 &c. the *sine* of 45°, gives 0·9659 &c. the *sine* of 75°.

211. The *sines* and *cosines* being found, the *tangents*, *cotangents*, &c. are obtained from similar triangles (see the fig. *Art.* 198):

Thus,

	AC	:	AO	::	CB	:	BG,
or	*cosine*	:	*sine*	::	*radius*	:	*tangent.* (204).
And,	AO	:	AC	::	CR	:	RP,
	sine	:	*cosine*	::	*radius*	:	*cotangent.*
Also,	AO	:	CO	::	BC	:	CG,
	cosine	:	*radius*	::	*radius*	:	*secant.*
And,	AO	:	CO	::	CR	:	CP,
or	*sine*	:	*radius*	::	*radius*	:	*cosecant.*

212. When the *sines*, *cosines*, &c. are computed to every minute up to 45°, and arranged in columns, they form a Table of the *natural sines*, *cosines*, &c. to every minute of the Quadrant: these are called *natural sines*, &c. because they exhibit the lengths in parts of the radius: i.e. the *Logarithms* of those numbers or natural sines, &c. compose the *artificial* or *Logarithmic Canon.*

Of the Table of Sines and Tangents.

213. The Table contains the Logarithms of the Sines and Tangents to every minute of the quadrant. Two degrees are in each page; and the minutes in the left, and right hand columns, answer equally for both.

The degrees up to 45 are at top, the minutes being in the left hand column; but the degrees from 45 to 90 necessarily fall in a contrary order at bottom, and the minutes are numbered upwards on the right.

Thus, if the arc or angle be 15° 17′ (page 32):

15° 17	*sine*	9·420933	.... the *cosine*	of 74° 43′.
	cosine	9·984363	.... *sine*	
	tang.	9·436570	.... *cotang.*	
	cotang.	10·563430	.... *tang.*	

214. But if the *radius* or *sine* of 90° be 1, its logarithm is 0·000000; and therefore as the *sine* of any other arc must, *in that case*, be less than 1, the index of its logarithm will be negative (166, *Arith.*). For example, when the radius is 1, the *sine* of 30° is $= \frac{1}{2}$, and the *cosine* or *sine* of 60° is = 0·8660254037 8 &c. (206). Now the logarithm of $\frac{1}{2}$ or 0·5 is — 1·698970; and the logarithm of 0·866025 &c. is — 1·937531; these are the *log. sine*, and *cosine* of 30° in the Table, excepting the indices, which, instead of — 1 and — 1, are 9 and 9.

If therefore, to avoid the use of negative indices in the logarithms (182, *Arith.*) we multiply, or suppose all the *sines*, &c. to be multiplied by 10000000000,

we shall get 0·5 × 10000000000 = 5000000000;

And the log. of 5000000000 is 9·698970, as in the table.

Also, 0·8660254038 × 10000000000 = 8660254038;

And the log. of 8660254038 is 9·937531, the tabular *cosine*.

The log. of the *radius* or *sine* of 90° will be 10·000000, which is the log. of 1 × 10000000000.

In like manner 0·0002908882 the *sine* of 1′ multiplied by 10000000000 gives 2908882, whose logarithm is 6·463726, the *log. sine* of 1′.

But the same indices will evidently result by considering the *sines*, &c. as computed to a *radius* of 10000000000 equal parts: Thus in the early printed tables of natural sines, tangents, &c. we find 127997841 the *tangent* of 44′, the *radius* being 10000000000: consequently 8·107203 the logarithm of 127997841, is the *log. tangent* of 44′.

215. The *log. secant* of an arc or angle is found by adding 10 to the index of the arithmetical complement of the *log. cosine.*

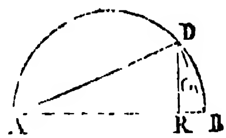
Thus, if the proposed angle be 30°:

Then (211),

As *cosine* of 30°	log.	9·937531	
		0·062469	*arith. comp.* (185, *arith.*)
to the *radius*	log.	10·000000	
so is the *radius*	log.	10·000000	
to the *secant*	log.	10·062469	

And the index of the arithmetical complement of the *log. sine* increased by 10 gives the *log. cosecant.*

216. To find the *log. versed sine* of an arc, add the log. of the number 2 to twice the *log. sine* of $\frac{1}{2}$ the arc, and the sum, rejecting 10 in the index, is the log. required.



For let ADB be a semi-circle, DR and RB the *sine* and *versed sine* of the arc DB: Then ADB, DRB being right angles, the triangles ADB, DRB will be similar, and we have

$$AB : DB :: DB : RB,$$

consequently DB is a mean proportional between AB and RB: and when the *radius* is 1, the diameter AB will be 2, therefore, if for AB and DB we substitute their measures, $\frac{DB^2}{2}$ is the value of RB.

Bisect DB in O; then DO or BO is the *sine* of $\frac{1}{2}$ the arc DB: and because the square on any line is equal to 4 times the square on $\frac{1}{2}$ that line, $4BO^2$ will be equal to DB^2; therefore $\frac{4BO^2}{2}$ or $2BO^2$ is the *versed sine* RB.

Suppose the arc DB = 30°, then BO is the *sine* of 15°:

15° *log. sine*	$9{\cdot}412996$	
	2	
	$18{\cdot}825992$	log. BO^2
2 log.	$0{\cdot}301030$	
Versed sine of 30° log.	$9{\cdot}127022$	

Sine RB : RD :: RD : RA; therefore RA, the versed sine of the supplement, is a third proportional to the *versed sine* and *sine* of an arc.

Let the arc DB = 30°. Then 30° *log. sine* (RD) 9·698970

	2
	19·397940
log. versed sine	9·127022
Suppl. versed sine log.	10·270918

217. If at any time it should be thought necessary to make use of a *log. sine* or *tangent* to parts of a *minute*, it may be found tolerably near by taking the proportional part of the difference of the *log. sines* or *tangents* next greater and next less (174, *Arith.*).

Thus, suppose the *log. tangent* of 17′ 20″ is required :

17′ *log. tang.*	7·694179	
18′ *log. tang.*	7·719003	
	24824	difference.

Then, as 60″ : 24824 :: 20″ : 8275 which added to the *log. tang.* of 17′ gives 7·702454 the *log. tang.* of 17′ 20″ nearly, the error being in defect because in this part of the Quadrant, the differences of the *log. tangents* in succession, decrease; for example, the difference of the *log. tangents* of 18′ and 19′ is less than that between the *log. tangents* of 17′ and 18′, &c.

And the foregoing operation reversed brings out the arc corresponding to a given *log. sine* or *tangent* :

Thus, to find the arc or angle answering to the *log. sine* 8·613714 :

given log.	8·613714	
next less	8·612563	*log. sine* 2° 31′
diff.	1151	

And the difference of the *log. sines* of 2° 31′ and 2° 32′ is 2865 :

Then, as 2865 : 60″ :: 1151 : 24″; therefore the angle is 2° 31′ 24″.

218. To find the *natural sine*, &c. corresponding to a given *logarithmic sine*, &c. when the *radius* is 1; take the number answering to the given logarithm from the table of the logarithms of the natural numbers; then the first figure on the left will be as many places to the right of units as the index is below 10; or as far to the left of units as the index is above 10 (214).

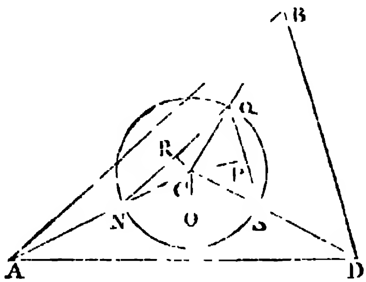
Thus, 7·241877 is the *log. sine* of 6′; and the number answering to the logarithm ·241877 is 17453, therefore 0·0017453 is the *natural sine* of 6′ to the *radius* 1.

Again, suppose we would find the *natural tangent* of 59° 21′.

The *log. tang.* is 10·228120, and the number to the log. ·228120 is 169091; now the index being 10, the first figure on the left will be an integer; therefore 1·69091 is the *natural tangent* of 59° 21′. In like manner, the *natural tangent* corresponding to the *log. tang.* 12·104901 is 127 321 &c.

219. The use of the sines in the resolution of Plane Triangles will appear from the following

THEOREM. *The sides of every plane Triangle are in the same proportion as the sines of their opposite angles.*



Let ABD be a triangle; C the centre of its circumscribing circle: then the radii CA, CB, CD being equal, the triangles ACB, ACD, BCD, are isosceles.

About C with any radius CN, describe a circle, and draw the chords NS, NQ, QS, which bisect with the perpendiculars CO, CR, CP; and the angles NCS, NCQ, QCS, will also be bisected (46, *corol.* 1).

And since the sides or radii CN, CQ, CS, are equal, the triangles NCS, NCQ, QCS will be isosceles and similar to ACD, ACB, BCD, respectively,

whence NS : AD :: NQ : AB :: QS : BD;

and NO : AD :: NR : AB :: QP : BD, because the halves of any lines must have the same proportion as the wholes.

But NO is the *sine* of the angle NCO; NR the *sine*

of NCR; and QP the *sine* of QCP to the same radius (204, *corol.* 1): And (71) the angle NCO is equal to NQS (or ABD); NCR equal to NSQ (or ADB); and QCP equal to QNS (or BAD). Therefore the sides AD, AB, BD, have the same proportion as NO, NR, QP, the *sines* of their opposite angles.

Thus if the angle A = 42°, B = 64°, D = 74°. Then the radius CQ, CS or CN being = 1,

NO = ·8988 &c. *sine* of 64° the angle B,
NR = ·9613 &c. *sine* of 74° angle D,
QP = ·6691 &c. *sine* of 42° angle A; and their doubles, or NS = 1·7976, NQ = 1·9226, QS = 1·3382 are the sides of the triangle NQS which is similar to the triangle ABD. Hence if one side of the triangle ABD be given, the other sides are found by proportion. Let DB (for example) = 100 *yards*:

Then QS : NS :: BD : AD,
viz. 1·3382 : 1·7976 :: 100 :
or ·6691 : ·8988 :: 100 : 134·3 *yards* nearly, by using the *sines* or halves of QS and NS, which have the same ratio as the wholes.

And QS : QN :: BD : BA,
or ·6691 : ·9613 :: 100 : 143·7 *yards* nearly, by taking the halves of QS and QN. But it is much more expeditious to work with the logarithms of the sines.

220. But independent of computation by the table of Sines, Tangents, &c. the several cases of Trigonometry are also resolved *geometrically*; and *instrumentally*. A scale of equal parts, with a Line of Chords or a Protractor for laying down or measuring angles, are sufficient for the *geometrical construction*, which is the most simple but least accurate method of solution.

The Sector is an instrument particularly adapted for trigonometrical operations. On each of its legs are laid down the natural sines, tangents, &c. together with the corresponding radius divided (on the 6-inch Sectors) into 100 equal parts: by those lines, the common proportions in trigonometry may be

wrought tolerably correct: But the Logarithmic or Gunter's Scale is the most commodious for that purpose. This Scale on the sector usually consists of three contiguous lines, namely, the line of numbers, that of sines, and the other of tangents, marked N, S, T; part lies on one leg, and part on the other, and therefore the sector must be quite open when it is used.

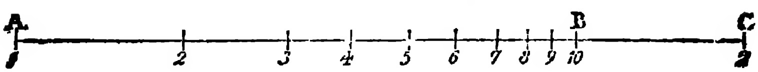
The Line of numbers is nothing more than the logarithms of the natural numbers from 1 to 10 taken from a scale of equal parts, and each extended from the beginning of the line on the left hand, towards the right: Thus,

From a scale of equal parts take ·301 the log. of 2, which set from 1 to 2.

And from the same scale set off ·477 the log. of 3, from 1 to 3:

And ·602 the log. of 4, from 1 to 4: and so on to 10.

Then the line AB will be the log. scale of numbers from 1 to 10, or from 10 to 100, or from 100 to 1000, &c,



And because the logarithms of 10 and 2 added together make the log. of 20, if the distance between 1 and 2 be set from B to C, then AC is the log. scale of 20 when AB is that of 10, or of 200 when AB is 100, &c.

But in taking the logarithms from a scale of equal parts, it is not necessary to consider them as decimals; for instead of ·301, ·477, ·602, &c. we may use any convenient numbers in the same proportion, as 3·01, 4·77, 6·02, &c. or, 30·1, 47·7, 60·2, &c. And when the scale is of sufficient length, these primary divisions may be divided and subdivided by laying off the logarithms of 1·1, 1·2, 1·3, &c. &c. as we find them on the 2 feet ruler called the Gunter's Scale.

In adapting the Sines and Tangents to the Scale of Numbers, the line AB is considered as the logarithm of the radius; for which reason the sine of 90° and the tangent of 45° are coincident with 10 (or B) on the scale. And when the sines and tangents correspond to a radius of 10, their logarithms are laid down from the left towards the right by means of the same scale of equal parts used for the logarithms of the natural numbers: Thus, the radius being 10, the sine of 30° is 5 (206), and therefore 30° on the line of sines answers to 5 on the line of numbers.

But because the radius is a mean proportional between the tangent and cotangent of an arc (204), it follows that the log. tang. and cotang. together always make double the log. of the radius or tang. of 45°, whence it is that the degrees above 45 on the line of tangents are numbered in a contrary order: thus 20° is also marked 70°; for the log. tang. of 70° is equal to the log. tang. of 45° together with the difference of the log. tangents of 20° and 45°. This inverted order of the tangents above 45° may be said to reduce the scale to half its length with the same extent of divisions.

Having premised what may be thought necessary respecting the Trigonometrical Canon, and the Logarithmic Scale; we shall now proceed to resolve the several Cases of Plane Triangles.

CASE I.

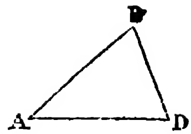
221. WHEN one side and the angles are given.

Examp. 1. Given AD = 360.

$$\text{Angles}\begin{cases} A = 43^\circ\ 15' \\ D = 72\ \ 51 \\ B = 63\ \ 54 \end{cases}$$

Required the sides AB and DB?

Geometrically.



From any convenient scale of equal parts, make AD = 360; then at the extremities A and D lay down the angle A = 43° 15′, and the angle D = 72° 51′ (144); produce AB and DB till they meet, and ABD is the triangle.

AB, and DB measured on the scale upon which AD was taken, will be found 380, and 275 nearly.

Arithmetically, or by computation.

By the Theorem *Art.* 219, As the sine of any angle,
Is to its opposite side,
So is the sine of any other angle,
To its opposite side.

The natural sines of to the radius 1 $\left\{\begin{array}{ll} 43° & 15′ \\ 72 & 51 \\ 63 & 54 \end{array}\right\}$ are $\left\{\begin{array}{l} 0·6852 \\ 0·9555 \\ 0·8980 \end{array}\right\}$ nearly, (218).

Therefore ·8980 (*sin.* ang. B) : 360 (AD) :: ·9555 (*sin.* ang D) : 383·1 nearly, = AB.

And ·8980 : 360 :: ·6852 : (*sin.* ang. A) : 274·7 nearly, = BD.

But the usual method by the logarithmic sines is much shorter: thus,

As the *sine* of the angle B, 63° 54	log. 9·953290
	0·046710 *arith comp.*(186, *Arith.*)
To the opposite side AD = 360	log. 2 556303
So is the *sine* of the ang. D, 72° 51′	log 9 980247
To its opposite side AB, 383·1	log. 2·583260

And

As *sine* 63° 54′	log. 9 953290
	0 046710
To AD	log. 2·556303
So is the *sine* of the angle A, 43° 15′	log. 9·835807
To the opposite side BD, 274·7	log. 2·438820

222. When the two first terms of the proportion are repeated, as in the present example, the operation may be somewhat abridged by taking the sum of the arithmetical complement and the log. of the 2*d*. term, instead of setting them down separately a second time;

Thus, 2·603013 is the sum of $\left\{ \begin{array}{l} 0·016710 \\ 2·556303 \end{array} \right.$

9·835807 *log. sine* 43° 15′
2·438820 log. of 274·7 as before.

Instrumentally, by the Logarithmic or Gunter's Scale.

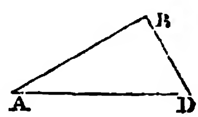
Set one foot of a pair of Compasses at 63° 54′ on the line of Sines and extend the other to 72° 51′, then that extent will reach the same way from 360 to 383 on the line of Numbers.

And the extent from 63° 54′ to 43° 15′ will reach from 360 to 275

The reason of this operation is evident from the nature of logarithms: for when 4 numbers are directly proportional, the second divided by the first, is equal to the fourth divided by the third, and *vice versâ* (22, *Arith.*); therefore the difference of the logarithms of the first and second terms is equal to the difference between the logarithms of the third and fourth (183, *Arith.*): Thus the difference of the log. sines of 63° 54′ and 43° 15′ is equal to the difference of the logarithms of 360 and 274·7.

Examp. 2. Given AD = 33·15.

Angles $\left\{ \begin{array}{l} A = 29° \; 0′ \\ D = 56 \; 11 \\ B = 94 \; 49 \end{array} \right.$



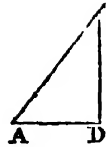
Required the sides AB and DB?

Here the first term of the proportions is the sine of 94° 49′ or 85° 11′ (200); and the sides will be, AB = 27·64, and BD = 16·13.

Examp. 3. Given AD = 1863.

Angles $\begin{cases} A = 49^\circ\ 17' \\ D = 90\ \ 0 \end{cases}$

Required the other two sides?



Construction. Take the base AD = 1863 from a scale of equal parts; and make the angle A = 49° 17′; then if DB be erected perpendicular to AD, the triangle is constructed.

Computation. Since the triangle is right-angled at D, the angle B is the complement of the angle A:

Therefore,

As *cosine* of angle A	log.	9·814160
		0·185540 *arith. comp.*
To AD, 1863	log.	3·270213
So *sine* of angle A, 49° 17′	log.	9·879677
To DB ....2164·7	log.	3·335390

And,

As *cosine* of the angle A		0·185540 *arith. comp.*
To AD	log.	3·270213
So is *sine* of angle D, 90°	log	10·000000
To AB 2856 ....log.		3·455753

By the Logarithmic Scale.

The extent from 40° 43′ to 49° 17′ on the line of sines, will reach on the line of numbers from 1863 to 2165 nearly, for DB.

And from 40° 43′ to 90° will reach from 1863 to 2855, the hypothenuse AB.

223. But the angle at D being a right one, the operation for finding the perpendicular DB is rather more simple by means of the *tangent* of its opposite angle A;

Thus,

As the *radius*	log.	10·000000
To the *tang* of the angle A, 49° 17′	log.	10·065178
So is AD, 1863	log.	3·270213
To DB, 2164·7	log.	3·335391

And the *secant* of 49° 17′ taken for the second term of the proportion, instead of the *tangent*, will bring out the side AE.

By the Log. Scale.

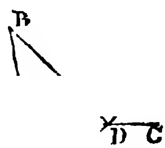
The extent from 45° to 49° 17′ (10° 43′) on the line of tangents (220) will reach (the contrary way) from 1863 to 2165 nearly, on the line of numbers.

CASE II.

224. WHEN two sides and an angle opposite to one of them are given.

Examp. 1. Given $\begin{cases} AB = 246{\cdot}5 \\ BD = 370{\cdot}5 \\ \text{Ang. } A = 101^\circ\ 21' \end{cases}$

Required AD, and the other two angles?



Construction. At A the extremity of an indefinite right line AC, make the angle CAB = 101° 21′, and set off AB = 216·5 from any convenient scale of equal parts; about B with BD = 370·5 taken from the same scale, describe an arc intersecting AC in D; draw BD; and ABD is the triangle.

The measure of the angle ADB is 41°, and that of B, 38°, nearly: and AD is 230 on the scale of equal parts.

Computation. The proportion in this case for finding an angle will be

As the side opposite the given angle,
Is to the *sine* of that angle,
So is the other given side,
To the *sine* of its opposite angle: Being the reverse of that in the former Case for finding a side.

As BD, 370·5	log.	2·568788
		7·431212 *arith. comp.*
To *sine* of angle A, 101° 21′	log.	9·991422
So is AB, 246·5	log.	2·391817
To the *sine* of the angle D, 40° 13′	log.	9·814451

Now the two angles A and D together make 142° 4′, therefore the third angle B is 37° 56′ (41).

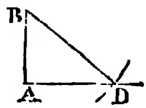
Then by *Case* 1:

As the *sine* of the angle ADB, 40°43′	log.	9·814451
		0·185549
To the opposite side AB.......	log	2·391817
So *sine* of angle B, 37° 56′........	log	9·788694
To AD, 232·3	log.	2·366060

By the Log Scale. The extent from 370½ to 245½ on the line of numbers, will reach from 78° 39′ (the supplement of 101° 21′) to 41° on the line of sines, for the angle ADB.

Examp. 2. Given $\begin{cases} AB = 49{\cdot}6 \\ BD = 81 \\ \text{Ang. } A = 90° \end{cases}$

Required AD, and the other two angles?



This is constructed in the same manner as the preceding example.

Computation.

As BD, 81	log.	1·908485
		8·091515
To *sine* of the opposite angle A, 90°	log	10·000000
So is AB, 49·6	log.	1·695482
To *sine* of the angle ADB, 37° 46′	log.	9·787297

And 52° 14′ the complement of 37° 46′ is the angle B.

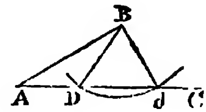
As the *sine* of the angle A, 90°	log.	10·000000
To BD	log.	1·908485
So is the *sine* of B, 52° 14′	log.	9·897908
To AD, 64·03	log.	1·806393

225. But AD may be found independent of the angles, thus (83, *corol.*):

Square of BD = 6561
of AB = 2460.16
diff. 4100·84, and its square root is 64·03 nearly, the side AD.

Examp. 3. Given $\begin{cases} \text{AB} = 4516 \\ \text{BD} = 2721 \\ \text{Ang. A} = 29^\circ\ 20' \end{cases}$

Required the other angles, and side?



Construction. Having made the angle A = 29° 20′, and AB = 4516, about B with 2721 describe an arc D*d* to intersect AC; draw BD, B*d* to the points of intersection; then either of the triangles ADB, A*d*B is *that* required.

For it is manifest that when the arc cuts the base line AC in two points, either AD, or A*d* will be the unknown side; and this ambiguity must always take place when the side (BD) opposite the given angle (A) is *less* than the other given side (AB), except the arc, instead of intersecting AC, should touch it; in which case the angle opposite AB becomes a right one (67, *corol.* 1). The single answers are therefore limited to examples where an angle opposite a given side is a right one, and such as have the side opposite the given angle greater than the other given side.

Computation.

As BD or B*d*, 2721 .... log. 3·434729
6·565271
Is to the *sine* of the opposite angle A, 29°20′ log. 9·690098
So is AB, 4516 .... log. 3·654754
To the *sine* of 54°24′ or its supplement 125°36′ log 9·910123

Therefore the angle ADB = 125° 36′ And A*d*B (BD*d*) = 54° 24′
ABD = 25 4 AB*d* = 96 16

Consequently,

As the *sine* of the angle A .... log. 9·690098
0·309902
Is to BD .... log 3·434729
So is the *sine* of the angle ABD, 25° 4′ log. 9·627030
To AD, 2353·2 .... log. 3·371661

And,

As the *sine* of the angle A		0·309902 *arith. comp.*
Is to Bd	log.	3·431729
So is the *sine* of the angle ABd, 96° 16′	log.	9·998397
To Ad, 5521·1	log.	3·740028

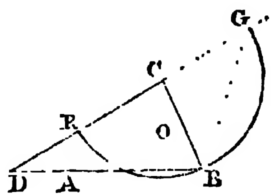
This is called the *ambiguous Case* in Trigonometry.

CASE III.

226. WHEN two sides and their included angle are given.

The two remaining angles will be found from the following *Theorem:*

As the sum of the given sides,
Is to their difference,
So is the *tangent* of half the sum of the two unknown angles,
To the *tangent* of half their difference.



Demonstration. Let DCB be the proposed triangle; CD, CB the given sides including the given angle DCB. Produce DC; and about C with the radius CB describe a semi circle: join BG, BR, and draw RO parallel to BD.

Now the sum of the angles CRB + CBR is equal to the sum of the unknown angles CDB + CBD, each sum being the supplement of the angle DCB to two right angles; Therefore as the triangle RCB is isosceles, each of the equal angles CRB, CBR, is equal to *half the sum* of the unknown angles CDB, CBD.

And because RO, BD are parallel, the angles RBD, BRO, will be equal, and the angle CRO equal to the angle CDB: But the angle RBD added to CBR (half the sum of the unknown angles) is the greater angle CBD; and the angle BRO taken from CRB (the like half sum) is the less angle CRO

(CDB); therefore BRO or RBD is *half the difference* of the unknown angles CBD, CDB *.

Let RA be parallel to BG. Then the angle RBG being a right one (72), BG and RA will be perpendicular to BR. Now if an arc was described about R with the radius RB, and another arc about B with the same radius, BG would be the tangent of the angle GRB or half the sum of the unknown angles; and RA the tangent of the angle ABR or half their difference.

But CG, CB, CR are equal, therefore DG is the sum, and DR the difference of the given sides CD and CB.

And because GB and RA are parallel, the triangles DRA, DGB will be similar; whence we have,

$$DG : DR :: GB : RA;$$

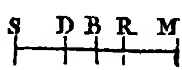
That is, as the sum of the sides, is to their difference, so is the *tangent* of half the sum of the unknown or opposite angles, to the *tangent* of half the difference of those angles.

Examp. 1. Let $CD = 4100$
$CB = 2265$
Angle $DCB = 87° 52'$.

Required the other two angles, and the side DB?

Construction. Make the angle $DCB = 87° 52'$; and from a scale of equal parts set off $CD = 4100$, and $CB = 2265$; join DB; and the triangle is constructed.

* Half the difference of any two numbers or lines added to, and subtracted from half their sum, give the greater, and less, respectively. Let BD, BR, be each equal to half the difference of two lines, and BS, BM, each equal to half their sum: then RS is the greater, and RM the less. For SM is the sum, and RD the difference of those lines.



DB measured on the same scale of equal parts is 4610 nearly.

And the measures of the angles D and B are 29° and 63° nearly.

Calculation.

CD = 4100			180° ′
CB = 2265	Included angle DCB	=	87 52
sum 6365	Sum of the unknown angles	=	92 8
diff. 1835	Angle CBR or CRB	=	46 4 half.

As 6365	log.	3 803798
		6·196 02
To 1835	log.	3·263636
So is the *tangent* of 46° 4′ (BRG)	log.	10·016174
To the *tangent* of 16·40 the angle RBA	log.	9·476012
Greater ang. CBD = 62·44 sum		
Less CDB = 29·24 diff.		

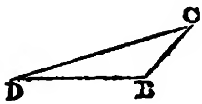
The side DB is found by *Case* I. thus,
As the *sin* of CBD, 62° 44′,
is to CD, 4100,
So is the *sine* of the angle DCB, 87° 52′,
To the opposite side DB, 4609·3.

By the Logarithmic Scale.

Having taken the extent from 6365, the sum of the sides, to 1835 the difference, on the line of numbers, set one foot of the compasses at 45° on the line of tangents, and let the other rest on that line while the foot which was on 45° is moved back to 43° 56′ (or 46° 4′); take off the compasses and set one foot on 45° again; then the other will extend to 16° 30′ nearly, the 4*th.* term of the proportion.

To explain this operation, it may be necessary to observe, that if the tangents above 45° were laid down on the scale in their natural order to the right of 45°, the extent from 6365 to 1835 would reach from 46° 4′ to 16° 30′ on the left; therefore the distance of 16° 30′ from 45° must be *less* than that extent by the distance from 45° to 43° 56′ (220); now the difference was found by moving one foot of the compasses from 45° while the other rested, and consequently that difference or extent when laid from 45° will give the 4*th.* term of the proportion, as in the last step of the process.

Examp. 2. Given $\begin{cases} CD = 94 \\ CB = 26 \\ \text{included angle } 22^\circ 20' \end{cases}$



Required the other angles, and the third side?

Answer. Angle D = 8° 2′
B = 149 38
DB = 70·7.

By the Logarithmic Scale.

CD =	94		180°	′
CB =	26	Included angle	22	20
Sum	120	2)	157	40
diff	68		78	50 half sum of unknown ang.

The extent from 120 to 68 on the line of numbers will reach from 78° 50′ (or 11° 10′) to 70° 30′ (or 19° 30) nearly, on the line of tangents. Here the extent from the first term of the proportion to the second is from right to left on the line of numbers, but the contrary way from the third to the fourth on the line of tangents, because (as it has been observed) the tangents above 45° are counted to the left.

Half sum of the unknown angles	78°	50
Half difference	70	30
Angle B	149	20 sum
Angle D	8	20 diff.

Now the extent from 149° 20′ (the angle B) to 22° 20′ (angle C) on the line of sines, will reach the same way on the line of numbers from 94 (DC) to 70, DB.

Examp. 3. Given $\begin{cases} BD = 22 \cdot 64 \\ BC = 36 \cdot 4 \\ \text{Angle } B = 90^\circ \end{cases}$

Required the angles at D and C, and the side DC?

Construction. Erect BC perpendicular to BD; then from a scale of equal parts (which should have a diagonal scale decimally divided) set off BD = 22·64, and BC = 36·4; join DC; and DBC is the triangle.

DC on the same scale measures 43.

And the angles D and C with the chords, will be found 58° and 32°.

Computation.

BC = 36·4
BD = 22·64
sum 59·04
diff. 13·76

As 59·04	log.	1·771146
		8·228854
Is to 13·76	log.	1·138618
So is the *tang*. of 45°, half the sum of angles D and C	log.	10·000000
To the *tang*. of 13 7′, half their difference	log.	9·367[illegible]

sum 58 7 angle D.
diff 31 53 angle C.

And DC found by *Case* I. is 42 86 &c.

By the Logarithmic Scale.

The extent from 59·04 to 13·76 on the line of numbers, will reach from 45° to 13° 10′ on the tangents, for half the difference of the angles D and C.

Half sum 45° 0′
Half diff 13 10
Ang. D 58 10
C 31 50

Then the extent from 31° 50′ to 90° on the line of sines, will reach from 22·64 to 43 nearly, for DC on the line of numbers.

227. But when the included angle is a right one, as in the present example, if either of the given sides be made *radius*, the other will be the *tangent* of its opposite angle (198). Therefore to find an unknown angle, suppose D,

As BD, 22·64	log.	1·354876
		8·615124
Is to BC, 36 4	log.	1·561101
So is the *radius*	log.	10·000000
To the *tang*. of 58° 7′, the angle D, as before;	log.	10·206225

By the Logarithmic Scale.

The extent from 22·64 to 36·4 on the line of numbers, will reach, on the line of tangents, from 45° to 58° 10′ (31° 50′) the angle D. For the 2*d.* term being greater than the first, the 4*th.* must be greater than 45°.

But the unknown side DC may be found without the angles, thus (83, *corol.*):

Square of BD = 512·5696
of BC = 1324·96

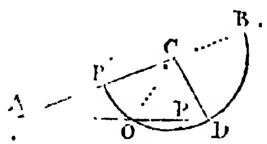
Sum 1837·5296; and the square root of this sum is 42·86 &c. the hypotenuse DC, as before.

CASE IV.

228. When the three sides are given.

We shall lay down two methods of finding the Angles.

1. Suppose ACD the proposed triangle; and let the perpendicular CP divide it into two right-angled triangles APC, DPC:



Then,

As the side AD,
Is to the sum of the other two sides AC, DC,
So is the difference of those sides AC, DC,
To the difference of PA and PD, the segments of the base AD.

Demonstration. Produce AC; and about C with the side CD describe a semi-circle, and draw CO. Then the radii CR, CD, CB being equal, AB is the sum of the sides CA and CD, and AR is their difference.

And because PO and PD are equal (65), AO will be the difference of the segments PA and PD: therefore (98),

AD : AB :: AR : AO which being taken from AD, and the remainder OD divided by 2, gives PD (or PO) one of the segments; and the sum of PO and AO is the other. Then the angles of the triangles APC, DPC are found by *Case* II.

Examp. 1. Let $\left.\begin{array}{l} AD = 462 \\ CA = 384 \\ CD = 169 \end{array}\right\}$ required the angles?

The Construction from a Scale of equal parts is according to *Art.* 136.

Calculation.

CA = 384
CD = 169
Sum 553 = AB
Diff. 215 = AR.

As 462 : 553 :: 215 : 257·35 = AO, nearly.
462 = AD.
Diff. 204·65 = OD.
Half 102·33 = PD or PO.
257·35 = AO.
359·68 = AP.

Now in the triangles APC, DPC,

are given AC = 384 and DC = 169
AP = 359·68 DP = 102·33

And the angles found by *Case* II. will

be PCA = 69° 30′ and PCD = 37° 16′
PAC = 20 30 PDC = 52 44

Therefore the angles are, C = 106° 46′
D = 52 44
A = 20 30

By the Logarithmic Scale.

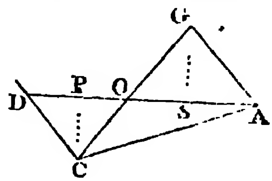
The extent from 462 to 553 on the line of numbers, will reach, the same way, from 215 to 257 nearly, on the same line.

The perpendicular however, may be drawn from either an-

gle; but when it falls without the triangle, it must meet the opposite side produced; in which case the calculation is no ways different from the preceding: Thus, suppose ACO to be the triangle; and let the perpendicular CP meet AO produced;

Then AO : AB (CA + CO) :: AR : AD; and half the difference of AD and AO is the segment PO as before: now the angles PCO, PCA being found (by *Case* II.), their *difference*, instead of the *sum*, will be the angle (ACO) opposite the base.

229. *Method* 2. This is principally derived from the preceding Demonstration. Thus, suppose CGA is the proposed triangle; and let it be required to find the angle CGA opposite the base CA.



Make GO = GA; and OC will be the difference of the sides GC and GA: then

As the rectangle of the sides GC and GA,
Is to the rectangle of half the sum and half the difference of CA and CO,
So is the square of the *radius*,
To the square of the *sine* of half the angle CGA.

Demonstration. Let AO produced meet CD drawn parallel to GA, and make GS and CP perpendicular to AD:

Then the triangles OCD, OGA will be isosceles and similar; and the angles OCD, OGA, and also the opposite sides are bisected by the perpendiculars CP, GS.

Now if AD is the base of the triangle ACD, and AC, DC, the other two sides, AO will be the difference of the segments PA, PD, exactly as in the preceding demonstration:

Therefore,

As the side AD,
Is to CA + CD, the sum of the other two sides,
So is CA − CD, the difference of those sides,
To OA.—Or because CO = CD, it will be

As AD : CA + CO :: CA − CO : OA:

And their halves will also be proportional,

or, As $\frac{1}{2}$ AD,
To half the sum of CA + CO,
So is $\frac{1}{2}$ the diff. of CA − CO,
To $\frac{1}{2}$ OA.

Therefore the rectangle $\frac{1}{2}$AD × $\frac{1}{2}$AO is equal to the rectangle under the $\frac{1}{2}$ sum and $\frac{1}{2}$ diff. of CA and CO (89).

But OP = $\frac{1}{2}$ OD, and OS = $\frac{1}{2}$ OA, therefore OP + OS or PS = $\frac{1}{2}$ AD; and consequently the rectangle PS × OS (= $\frac{1}{2}$ AD × $\frac{1}{2}$ AO) is equal to the rectangle of the aforesaid $\frac{1}{2}$ sum and $\frac{1}{2}$ difference.

Now the triangles OPC, OGS being similar, we have
OC : OG :: OP : OS; and by composition (94, *schol.*)
OC + OG (GC) : OG :: OP + OS (PS) : OS,
or GC : OG :: PS : OS:

And GC × OG : OG × OG :: PS × OS : OS × OS, by taking equimultiples of the two first terms of the proportion, and also of the two last (Arith. 95):

or GC × OG : OG^2 :: PS × OS : OS^2:
whence GC × OG : PS × OS :: OG^2 : OS^2.

But PS × OS is = the rectangle under the $\frac{1}{2}$ sum and $\frac{1}{2}$ difference of CA and CO, hence the last proportion becomes

As GC × OG, or GC × GA,
Is to the rectangle of $\frac{1}{2}$ the sum and $\frac{1}{2}$ the diff. of CA and CO,
So is OG^2, to OS^2:

Therefore, if OG or GA be made the *radius*, OS will be the *sine* of the angle OGS, or of half the angle CGA.

Example 2 Let the sides of the Triangle CGA be as in the preceding example, namely, CA = 462, GC = 384, GA = 169.

Then,

GC = 384			
GA = 169			
215 diff.		7·415669	*arith. comp.* log 384 = GC.
CA = 462		7·772113	*arith. comp.* log. 169 = GA.
sum 677	half = 338·5 log.	2·529559	(186, *Arith.*)
diff. 247	half = 123·5 log.	2·091667	
	radius square, log.	20·000000	
	2)	19·809008	
Angle OGS 53° 23′	log. *sine*	9·904504	

Angle CGA . 106 46 as before.

The other two angles are found by *Case* II.

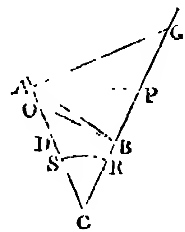
The method of working the last proportion by the Logarithmic Scale is omitted, it being rather complex, and therefore may produce considerable uncertainty in the results, particularly on the six-inch Sectors. We may also remark in general respecting these operations, that when the sides of the triangles exceed 1000, the calculations should be made with the pen, because there is too much *guess-work* on the Scales when the integers are more than three.

Application of Trigonometry to measuring Heights and Distances.

230. THE Instrument proper for measuring horizontal and vertical angles in common Trigonometrical operations is a Theodolite furnished with one or two Telescopes, and a Vertical arc: And if the horizontal circle is not less than about $6\frac{1}{2}$ inches in diameter, the observed angles may be read off to half a minute. The student, however, would benefit little from a

description of the Instrument, because the method of examining and correcting the Spirit-levels, &c. and adjusting the whole for observation, must be acquired under the eye of a Master.

But after all the care that may have been bestowed in correcting the line of collimation, telescope-level, &c. it seldom happens that the elevations or depressions shown by the Instrument are correct. It is therefore always adviseable to determine the *error*, or how much the elevations or depressions are too great, or too little. This may be done in the following manner:



Let C be the centre of the earth, SR an arc on its surface, A the place of the telescope when the Theodolite stands in the vertical line CA, B the place of the telescope when it stands in the vertical CB, AG (perpendicular to AC) the horizontal line at A drawn to meet CG, and BO (at right angles to BC) the horizontal line at B.

Then, if the telescope at B be directed to a mark or object at A, the elevation of that object above the horizontal line BO is the angle OBA; and when the telescope is at A, and directed to an object at B, its depression below the horizon AG will be the angle GAB.

Let SD = RB, and RP = SA. Then because the triangles APC, DBC are isosceles, and the angles CAG, CBO right ones, the angle CAP + angle PAG = a right angle; but the angle CAP + half the angle ACP also make a right angle, therefore the angle PAG or its equal DBO, is equal to half the angle C.

Now the depression or angle GAB = GAP + PAB (or ABD); or GAB = PAG + DBO + OBA; but PAG + DBO = angle C:

Therefore the depression GAB = ang. C + *elev.* OBA; or *depr.* GAB + *elev.* OBA = ang. C + twice the *elev.* OBA; Therefore the elevation and depression together, lessened by the angle C, is equal to twice the elevation: consequently *half the difference between the sum of the elevation and depression, and the angle C, is the elevation.*

Now, whatever be the error in elevation or depression, their *sum* will be constant; for one is always diminished by the same quantity that the other is augmented; hence the preceding rule gives the *true elevation*, except the angle C be greater than the elevation and depression together, in which case, the said *half difference* is the *true depression* of the highest of the two points or objects A, B.

And when the observations are both elevations, or both depressions, their *difference* is constant, and *half the difference between the angle* C *and that constant difference will be the true elevation of the highest of the two points* A, B, *if the angle* C *be the less, but equal to the true depression of that highest point or object, when it is the greater.*

Should both the reciprocal observations be depressions (or both elevations), and equal to each other, the vertical heights SA, and RB are equal; and the true depressions will be half the angle C.

Examp. The following observations were made with a Theodolite for determining the error in the vertical angles taken with that instrument.

Two marks, A and B, were set up exactly at the same height above the ground as the height of the telescope; and at A, the depression of B, or the angle GAB was 24'; and at B, the elevation of A, or the angle OBA = 12'. The distance of the stations or arc SR was 2600 *yards*, which, allowing 69½ miles to a degree, gives 1'·28 of a degree nearly, the angle C.

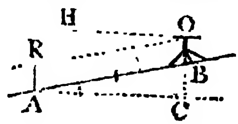
Then, $\frac{24' + 12' - 1'\cdot 28}{2} = 17'\cdot 36$ or about $17'\frac{1}{3}$ the true elevation or angle OBA; consequently $17'\frac{1}{3} - 12' = 5'\frac{1}{3}$ is the *error*, or what the altitudes shown by the instrument were too little, or the depressions too great.

A distance of 600 or 700 *yards* however, is sufficient for trying a common Theodolite. In which case the angle C may be neglected, and the verticals SA and RB considered as parallels: the expressions then become more simple. Thus if one observation be an elevation $= 17'$, and the other a depression $= 13'$, then *half their sum* $= 15'$ is the true elevation or depression; and $17' - 15' = 2'$ is what the instrument gives elevations too great.—If both are elevations, or both depressions *half the difference* is the true elevation of one station, and the true depression of the other.

Here the observations themselves are supposed to be correctly made; for the result will evidently partake of any error that may arise in consequence of a mistake.

231. Short Bases for temporary use only, are sometimes measured with Rods, or the Gunter's Chain of 66 feet. But the common 50, or 100 feet Tapes are much better adapted for expedition: with these lines, when the ground is tolerable level, and the direction or *alignement* of the base pretty correct, the error in distance will probably be about 3 inches in 50 feet, or $\frac{1}{200}$ of the whole measurement as long as the Tapes are kept dry: after frequent use however, they should be tried on a level pavement, or long floor, for which purpose a distance of 50 feet may be laid down by means of one or more Rods properly adjusted in respect of length.

232. When a Base is measured on sloping ground, it must be reduced to the corresponding horizontal line, if horizontal angles at its extremities are taken with a Theodolite. Suppose AB is a base of 300 *yards*; OB a Theodolite; and let the height of the staff AR be equal to OB the height of the instrument; also suppose HOR, the angle of depression of the top R below the horizontal line HO is 5°; then if OC is perpendicular to HO, the line AC, parallel HO, will be the horizontal base corresponding to the measured base AB.



Now the angles HOR, BAC being equal, we have (by *Case* I.)

As *radius*	log.	10·000000
To AB, 300	log.	2·477121
So is *cosine* of 5° (the angle BAC)	log.	9·998344
To AC, 298·9	log.	2·475465

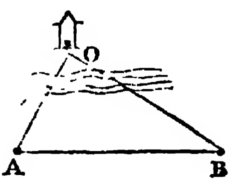
The difference of AB and AC is only 1·1 *yds.* Therefore a reduction of this kind seems unnecessary when the measured base is inclined to the horizon in a small angle, except the operation is intended to produce a very accurate result.

233. EXAMPLES.

1. To find the distances AO, BO from the stations A and B to the inaccessible object O, I measured AB which was 730 *feet*, the ground being nearly level; and having set up marks at A and B, the angles at those stations, taken with the Theodolite

were $\begin{cases} A = 57° 12', \\ B = 24\ 45. \end{cases}$ Whence the distances AO, BO are required?

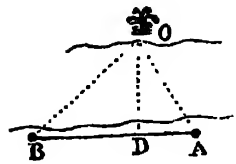
The angle at O, or supplement of the angles A and B is 98° 3'. And the *Construction* and *Calculation* will be exactly the same as in the two first examples, *Case* I. (221).



And $\begin{cases} \text{AO will be found} = 308{\cdot}6 \textit{ feet}, \\ \text{BO} \qquad\qquad\qquad = 619{\cdot}7 \end{cases}$

2. Wanting to know the breadth (DO) of a river, I measured a base AB of 100 *yards* along the bank, and at the extremities A and B took angles to an object O on the opposite side,

namely $\begin{cases} \text{angle OBA} = 37° 40', \\ \text{angle OAB} = 59\ 15. \end{cases}$ Hence the breadth OD is required?



Construction. Make BA = 400 from any convenient scale of equal parts; and at the extremities B and A, lay down the respective angles 37° 40′ and 59° 15; then the perpendicular OD upon the base BA (152), will be the breadth required. And its measure is 212 nearly.

Calculation. By *Case* I. (221).

As the *sine* of the angle BOA, 83° 5′ (the supplement of the angles B and A)	log.	9·996828
		0·003172
Is to BA, 400	log	2·602060
So is *sine* of angle B, 37° 40′	log.	9·786089
To AO	log.	2·391321

Then,

As *sine* of the angle ODA, 90°	log.	10·000000
Is to AO	log.	2·391321
So is the *sine* of the angle A, 59° 15′	log	9·934199
To OD, 211·5 *yards*	log.	2·325520

By the Logarithmic Scale. The extent from 83° 5′ to 37° 40′ on the sines, will reach from 400 to 245 (AO) on the line of numbers.

Then, the extent from 90° to 59° 15′ on the sines, will reach from 245 to 210, for OD, on the line of numbers.

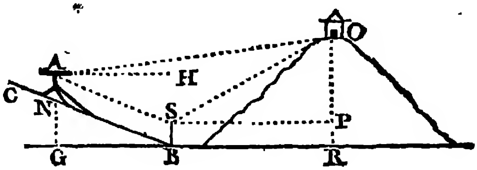
3. To find the height, and the distance of the object O on the top of a hill from the station B, we measured a base BN of 642 *yards* up the sloping ground BC, directly from the object O, the points O, B, N, being in the same vertical plane, then having set up a staff BS whose length was equal to the height of the Theodolite, we found the angles of elevation and depression to be as follows:

At the station N, { object O elev. 3° 59′ = ang. OAH,
top of staff S depr. 39′ = ang. HAS.

At the other station B, the elev. of O = 5° 52′ = ang. PSO.

Hence the horizontal distance BR, the height RO, and also GN the height of the station N above B, are required?

Method of Construction. Draw RG indefinitely to represent an horizontal line, and from any point B draw the slope BC making the angle CBG = 39′ (the angle HAS): then from a scale of equal parts set off BN = 642, and make BS perpendicular to BG and equal to the height of the Theodolite NA; let SA be parallel to BC and equal to BN, and AG parallel to SB; also draw the horizontal lines, AH, SP: then if the angles OSP, OAH are made equal to 5° 52′, and 3° 59′, the angles of elevation respectively, and OR is perpendicular to GR, the figure will be constructed.



Calculation.

Angle OAH = 3° 59′
HAS = 39 .... its supplement 179° 21′ angle ASP
Angle OAS = 4 38 subtract 5 52 angle OSP
173 29 angle OSA

Therefore the angles of the triangle OAS are OSA = 173° 29′
OAS = 4 38
AOS = 1 53

By *Case* I. (221).

As *sine* AOS, 1° 53′ .... log. 8·516726
1·483274
To AS. 642 .... log. 2·807535
So is *sine* of OAS, 4° 38′ .... log. 8·907297
To SO .... log. 3·198106

Then, as *sine* SPO, 90° .... log. 10·000000
To SO .... log. 3·198106
So is *sine* OSP 5° 52′ .... log. 9·009515
To the height OP, 161·3 .... log. 2·207621

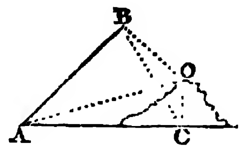
SO ..... log. 3·198106 (222)
5° 52′ *cosine* .... 9·997719
Distance SP = BR = 1569·7 log. 3·195825

As *sine* ang. NGB 90° .... log. 10·000000
To NB 642 .... log. 2·807535
So *sine* NBG 39′ .... log. 8·054781
To NG 7·3 *yards*, nearly .... log. 0·862316

And if SB (PR) the height of the Theodolite when standing on the ground, be added to OP, we shall have the height of O above the horizontal line GR.

N. B. If a correct result is required from an operation of this kind, the error (if any) in angles of elevation should be determined (230); and care must be taken to adjust the height of the instrument when at B, so that the telescope may be exactly at the height BS from the ground.

4. Wanting to know the distance (AC), of a hill from the station A, and also the height (OC); we measured a base AB of 298 *yards* on ground nearly horizontal, and at the extremities A and B observed the horizontal angles, BAO (or BAC) = 42° 17′, ABO (or ABC) = 79° 29′; and at A the angle of elevation OAC was 4° 51′. Required the distance AC, and height CO?



Method of Construction. The three points A, B, C being supposed in a plane parallel to the horizon, and the plane of the instrument at A and B in that plane, the angles taken to the point O in the perpendicular CO will be the same as they would be if the telescope was directed to the point C, because the horizontal circle of the Theodolite is not moved by elevating or depressing the telescope. Therefore, having made AB = 298, and the angle BAC = 42° 17′, ABC = 79° 29′, and OAC = 4° 51′, raise the perpendicular CO; then AC is the distance, and CO the height sought.

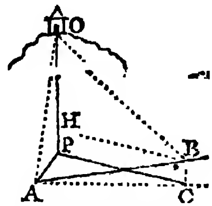
Calculation. The angle ACB is 58° 14′ the supplement of the horizontal angles at A and B.

As *sine* of 58° 14′	log	9·929521
		0·070479
To AB, 298	log.	2·474216
So is *sine* of ABC, 79° 29′	log.	9·992613
To AC, 341·6	log.	2·537338
Ang. OAC = 4° 51′	*tang.*	8·928658 (223)
Height CO = 29·2	log.	1·465996

And the height of the instrument being added to 29·2 *yards* will give the whole height of the top O.

5. To find the distance of the object O on the top of a hill from the station A, and also its height, we measured a base AB of 210 *yards* up sloping ground, its inclination with the horizontal line AC being 9° 30′ the angle BAC; and the horizontal angles at A and B (found by directing the telescope to O) were PCA = 64° 10′, and PAC = 76° 17′; also the angle of elevation OBH (HB being an horizontal line) was 5° 34′. From hence the height, and distance of the object O are required?

Method of Calculation. Let the horizontal lines BH, AP meet OP the line from O perpendicular to the horizon; and suppose AC is the horizontal base (232), and BC perpendicular to AC.



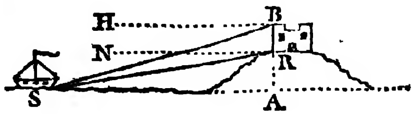
In the right angled triangle ACB, the hypotenuse AB and all the angles are given, whence CB the height of the station B above A will be found = 34·7 *yards*: and AC the reduced base = 207·1 *yards*.

Then AC, and the angles of the triangle ACP, $\left\{\begin{array}{l} PCA = 64° \ 10' \\ PAC = 76 \ \ 17 \\ APC = 39 \ \ 33 \end{array}\right.$ will give AP the horizontal distance from A = 292·8 *yards*; and CP the horizontal distance from C = 316 *yards* = BH.

Now in the right angled triangle OHB the side BH and all the angles are given, whence HO = 30·8 *yards* the height of O above B, to this add BC and we have OP = 6[illegible] *yards* the height above A. To this also should be added the height of the instrument for the whole height of O above the ground at A.

6. At B, the top of a castle which stood on a hill near the sea shore, the depression of a ship at anchor was 4° 52′ (the angle HBS), and at R, the bottom of the castle, its depression was 4° 2′ (the angle NRS). Required the horizontal distance of the vessel, and also the height of the castle above the level of the sea, supposing RB the castle itself to be 54 feet high?

Method of Construction. From any scale of equal parts make BR = 54, and draw the horizontal lines RN, BH at right angles to BR: let the angles HBS, NRS be made = 4° 52′, and 4° 2′, respectively; then if SA is drawn perpendicular to BR produced, it will be the horizontal distance, and AR the height of the bottom of the castle.



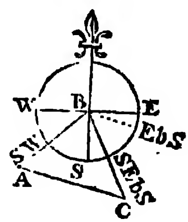
Method of Calculation. The angle BSR is equal to 50′ the difference of the angles of depression, therefore by *Case* I. (221).

As the *sine* of 50′
Is to BR, 54,
So is the *sine* of the angle SBR (the *cosine* of 4° 52′),
To SR.

And as the *sine* of ang. A, 90°
To SR,
So is *sine* of ang. RSA (NRS) 4° 2′,
To AR. 260 *feet*.
And, so is *cosine* of 4° 2′, to 3690 *feet* = AS the horizontal dist.

7. In surveying with the compass, an object C bore SE b S, and when we had gone 240 *yards* in a SW direction, the object bore E b S. Required its distance from the stations B and A?

Construction. Let the circle whose centre is B represent the compass; E, W, S, the east, west, and south points; draw E b S one point or 11¼ deg. from E; S E b S three points or 33¾ deg. from S; and SW four points or 45° from S; and make BA = 240 from a scale of equal parts; then if AC be drawn parallel to the E b S direction, C will be the place of the object.



Method of Calculation.

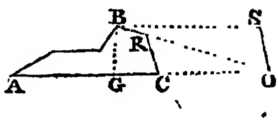
In the triang. ABC $\begin{cases} \text{ang. ABC} = 7 \textit{ points} \text{ or } 78^\circ\ 45' \\ \text{ACB} = 4 \textit{ points} \text{ or } 45 \\ \text{BAC} = 5 \textit{ points} \text{ or } 56\ \ 15 \end{cases}$

And the side BA = 240, whence, by *Case* I. AC = 333, and BC = 282 *yards*.

8. If BG the height of the rampart ABRC be 16 *feet*, and the exterior talus BR of the parapet is inclined to the horizon in an angle of 4° ; what is the difference in the distance (BO) of a musket shot made directly in front, and another (BS) inclined to that direction in an angle (OBS) of 40°, both shots being made in the plane of the talus?

Calculation.

As *sine* of ang. GOB, 4°	log.	8·843585
		0·156415
To GB 16	log.	1·204120
So *sine* of ang. G 90°....	log.	10·000000
To BO, 229·4	log.	2·360535



Now OS is the intersection of the plane of the horizon and that of the talus, therefore the direct shot, or the line BO is at right angles to OS, and consequently the angle BSO is the complement of OBS;

Whence, as *cosine* of 40°	log.	9·884254
		0·115746
To BO	log.	2·360535
So *sine* of BOS, 90°.......	log.	10·000000
To BS = 299 4	log.	2·176281

BO = 229·4

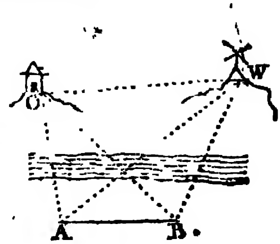
dif. 70 *feet, Ans.*

9. Wanting to know the horizontal distance between the inaccessible objects O, W, and also their heights, we measured a base AB of 670 *yards* on ground nearly horizontal; and at the extremities A and B took the following angles:

At A, ang $\begin{cases} \text{BAW} = 40° \ 16'. \\ \text{WAO} = 57 \ \ 40. \end{cases}$ Elevation of W = 3° 46'. of O = 3 33.

At B, ang. $\begin{cases} \text{ABO} = 42° \ 22' \\ \text{OBW} = 71 \ \ 7. \end{cases}$

Hence the heights, and distance OW are required?



Construction. From any convenient scale of equal parts, make AB = 670, and lay down the respective horizontal angles at A and B; join O, W, the points of concourse of the lines from A and B; and OW measured on the scale from which AB was taken, will be 1170 *yds.* nearly, the distance between the objects.

Calculation.

The angles of the triangle AOB are $\begin{cases} ABO = 42° \; 22' \\ OAB = 97 \;\; 56 \\ AOB = 39 \;\; 42 \end{cases}$

Whence, by *Case* I. (221), AO will be found = 706·8 *yards.*

And the angles of the triangle AWB $\begin{cases} BAW = 40° \; 16' \\ ABW = 113 \;\; 29 \\ AWB = 26 \;\; 15 \end{cases}$

will give AW = 1389·4 *yards.*

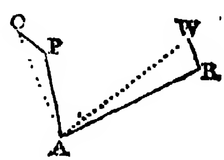
Now in the triangle OAW we have given $\begin{cases} AW = 1389·4 \\ AO = 706·8 \end{cases}$

and the included angle OAW = 57° 40′.

To find OW (by *Case* III.)

		AW =	1389·4		
		AO =	706·8		
	180° 0′	sum	2096·2	.... log.	3·321433
	57 40				6·678567
sum of unknown ang.	122 20	diff.	682·6	.... log.	2·834166
half	61 10			.... *tang.*	10·259233
half diff.	30 36			.... *tang.*	9·771966
ang. OWA =	30 34				

Then, as *sine* OWA, 30° 34′	log	9·706326
		0·293674
to AO	log.	2·849310
so is *sine* ang. OAW = 57° 40′	log.	9·926831
to OW, 1174 *yards*	log.	3·069815

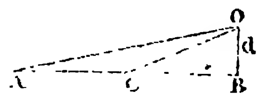


The lines AW, AO in the preceding figure are represented as drawn to the objects W and O on the elevated situations, because the telescope of the Theodolite is pointed to them when the angles are observed; but the distances by *construction* and *calculation* are the *horizontal lines*

supposed to be drawn from the stations to meet the perpendiculars OP, WR, let fall from O and W upon the plane of the horizon. (233, *Examp.* 4.)

Therefore to find the heights of O and W, we have, in the triangles ARW, APO right angled at R and P, the distances AR, AP equal to 1389·4, and 706·8 *yards*, and the angles at A, 3° 46′, and 3° 33′ the elevations, whence (223) WR will be found = 91·5, and OP = 43·9 *yards*.

10. When a distant object near the horizontal plane subtends a small angle, the following method of determining its distance would be simple, could we measure such angles with accuracy and expedition. Let OB be a distant object, and suppose the angles OAB, OCB, are 2′ and $2\frac{1}{4}'$, respectively, the base or distance AC, which is in a direct line from A towards the object, being 400 *yards*. Let C*d* be parallel to AO; then the triangles BC*d*, BAO will be similar, whence



$$BO : Bd :: AB : CB, \text{ and by division, } (91, \textit{corol. } 3.)$$
$$BO - Bd : Bd :: AB - CB : CB.$$

But the angle BC*d* is equal to the angle BAO; and because the sines or tangents of small arcs are nearly in the same proportion as the arcs or angles themselves (208), BO and B*d* will be as the opposite angles BCO, BC*d*, therefore the proportion becomes $2\frac{1}{4}' - 2 : 2' :: 400 : 3200$ *yards*, the distance CB: That is, as the difference of the angles, is to the less angle, so is the difference of the distances or measured base, to the less distance.

Corol. Hence the distances BC, BA, are reciprocally as the angles subtended at A and C.

Remark. Several attempts have been made to bring this method into general practice; and some ingenuity displayed in contriving instruments for measuring the angles; but it is known from experience that the extremities or boundaries of objects

standing on the ground at any considerable distance, seldom appear, even through good telescopes, sufficiently defined to permit the angles to be taken to that precision which is evidenly necessary when a satisfactory result is required; for a small error in either angle will produce a very considerable one in the distance. Thus, in the foregoing example, suppose an error or variation of 3″ in the angle OCB, or let it be 2′ 12″ instead of 2′ 15″,

Then, as 12″ (the difference of 2′ 12″ and 2′), is to 2′, so is 400 to 4000 *yards* the distance CB, instead of 3200.

Again, Let the base AC = 300 *yards*, and suppose the angles at A and C are 3′ 20″ and 4′, respectively;

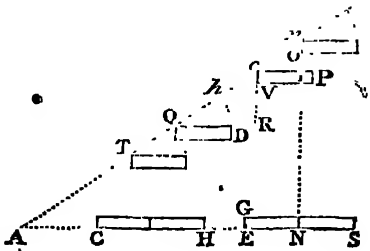
Then, as 40″ (their difference), is to 3′ 20″, so is 300 *yards*, to 1500 *yards* = CB.

Now admit an uncertainty of 3″ in each angle, and take that at A = 3′ 23″, and the other at C = 3′ 57″, and we have

As 34″ (the difference) : 3′ 23″ :: 300 *yards* : 1794 *yards*; the difference is 294 *yards* in about a mile; an uncertainty perhaps as great as that in an estimate by the eye at the same distance.

11. If CS be a line of cavalry; to determine the wheeling intervals between half squadrons marching *en echellon* from the right, when halted and formed on the line AB which is inclined to CS in a given angle

Construction. Let CS consist of two squadrons CH and ES; the extent of each = [illegible] *yards*, depth CE = 7½ *yards*, the interval HE = 16 *yards*, and suppose the angle BAS = 35°.



Drawn N*n* perpendicular to CS (the half squadron NS being supposed to march from N to O in a direction perpendicular to CS),

and make $ne = NE$, $eh = EH$; lQ, QT each $= NE$; and from n, e, Q, T, draw lines parallel to CS, and on those lines make parallelograms each equal to GN for the half squadrons: then if the half squadrons wheel on the pivots n, e, Q, T, till their fronts are in the line AB, the extent TB will be equal to CS, with the proper interval between the squadrons, or $he = HE$.

Calculation. We want the perpendicular distances OI, IP, and VR, DR.

$$en = EN = 24 \textit{ yards} = 72 \textit{ feet.}$$
$$eQ = NH = 40 \textit{ yards} = 120 \textit{ feet.}$$

In the right angled triangles eIn, QRe the angles at Q and e are $35°$.

As *rad.* : 72 ($=en$) :: *sin.* $35°$: 41·3 *feet* $= nI$, whence OI $=$ 19 *feet*, nearly.
rad. : 72 :: *cosin.* $35°$: 59 *feet* $= eI$, whence IP $=$ 13 *feet*, nearly.
rad. : 120 ($=eQ$) :: *sin.* $35°$: 68·8 *feet* $= eR$, whence VR $=$ 46 *feet*, nearly.
rad. : 120 :: *cosin.* $35°$: 98 *feet*, nearly, whence DR $=$ 26 *feet*.

But the measurement of these lines from construction, will be sufficiently correct for practical purposes.

234. In the preceding examples, the angles subtended by distant objects are supposed to be in an horizontal, or in a vertical plane: We shall now give the method of computation when they are measured in planes oblique to the horizon.

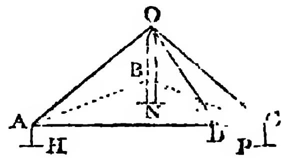
Angles oblique to the horizon are usually taken with a sextant or Hadley's quadrant, which is held in a position so that its plane passes through both objects and the eye of the observer. And elevations are found by reflecting the object from an artificial horizon. But whoever intends to observe with a sextant must acquire the method of using it from *practice* under the direction of a person who is master of the several adjustments, &c.; for which reason we shall not attempt a description of the instrument.

EXAMPLES.

1. Suppose ON is an object standing on the horizontal plane HNP; HA and PC two staves or rods equal in height to that

of the eye; and let the plane ABC be parallel to the horizontal plane HNP; also suppose HP or AC is a base of 250 *yards*; and that the angles taken in the plane OCA, are OAC = 56° 46′, and OCA = 62° 54′; the angles of elevation OAB, OCB being 6° 40′, and 7° 6′, respectively. Hence the height, and horizontal distances AB, CB, are required?

When one of the sides (AC) including an angle (OAC) oblique to the plane of the horizon, is horizontal, the angle is reduced to the corresponding horizontal angle by the following proportion,



As the *cosine* of the angle of elevation (OAB),
Is to the *cosine* of the given ang. (OAC),
So is the *radius* or *sine* of 90°,
To the *cosine* of the reduced angle (BAC).

For let DBO be a vertical plane and the angle ADO a right one; then the triangles ABO, DBO being also right angled at B, we shall have, (*Case* I. 221)

Sine ABO, 90° : AO :: *sine* AOB : AB;
Sine ADO, 90° : AO :: *sine* AOD : AD;

Therefore by equality,

sine AOB : *sine* AOD :: AB : AD :: *sine* ADB, 90° : *sine* ABD;

or, *sine* AOB : *sine* AOD :: *sine* 90° : *sine* ABD:

But AOB is the complement of the elevation; AOD the complement of the observed angle OAC; and ABD that of the reduced angle BAC; therefore, &c.

As *cosine* 6° 40′	log	9.997053
		0.002947
To *cosine* 56° 46′	log.	9.738820
So *sine* 90°	log.	10.000000
To *cosine* 56° 31′ the reduced angle BAC	log.	9.741767

As *cosine* 7° 6′	log.	9·996657
		0·003343
To *cosine* 62° 54′	log.	9·658531
So *sine* 90°	log.	10·000000
To *cosine* 62° 40′ the reduced angle ACB		9·661874

Therefore the angles of the triangle AOC reduced to the horizontal plane are $\begin{cases} BAC = 56° \ 31' \\ ACB = 62 \ \ 40 \\ ABC = 60 \ \ 49 \end{cases}$

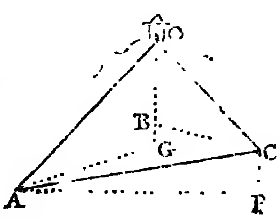
And the side AC being 250 *yards*, we shall have (by *Case* I. 221) AB = 254·4, and CB = 238·8 *yards*; whence BO = 29·7 *yards*: to this add NB the height of the observer's eye above the horizontal plane HNP, and the sum will be the whole height NO.

But the distances AB, CB, and height BO may be calculated without any reduction of angles; for AC and all the angles of the triangle AOC being given, the sides AO, CO are found by *Case* I. and then the right angled triangles ABO, CBO, will give AB, CB, and BO at three proportions.

And should it be necessary, the reduced angles may be found from the sides of the triangle ABC, by *Case* IV (228).

2. If A and C are two stations on sloping ground; O an object on the top of a hill, and the angles OCA, OAC (measured with a sextant) equal to 79° 29′, and 63° 11′, respectively; also suppose the angle of elevation at A is = 6° 36′, at C = 5° 22′: What are the horizontal distances and height of the object, AC being = 410 *yards*?

Let OG be perpendicular, and AG, CB, parallel to the horizon: then AG, CB are the horizontal distances.



In the triangle AOC the angles are $\begin{cases} OCA = 79° \ 29' \\ OAC = 63 \ \ 11 \\ AOC = 37 \ \ 20 \end{cases}$
And AC = 410 *yards*.

Whence (221) AO = 664·7, CO = 603·4, these hypotenuses, with the angles of elevation OAG, OCB, in the right angled triangles AGO, CBO, give AG = 660·3, OG = 76·4, CB = 600·7, OB = 56·4 *yards*.

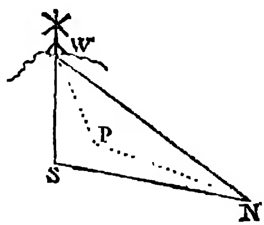
And the difference of OG and OB is 20 *yards* = BG = CP the difference in the heights of the stations, AP being supposed horizontal.

The sides AC, CP, will give AP. And the angles of the triangle AOC when reduced to the horizon, may be found from the horizontal distances AP, AG, CB, taken as the sides of a triangle (228).

3. At a mile-stone N on the ascending road NS we observed the angle SNW between the next mile-stone S and the windmill W on the top of a hill, and found it to be 46° 37′; the elevation of W, or angle WNP was 3° 49′; next, at the mile-stone S, the angle NSW measured 91° 4′. Hence the horizontal distance NP, and height PW are required?

The angles of the triangle SWN are $\begin{cases} SNW = 46^\circ 37' \\ NSW = 91 \quad 4 \\ SWN = 42 \quad 19, \text{ and} \end{cases}$
NS = 1760 *yards*:

these give NW = 2614:



Then in the triangle WPN, right angled at P, the hypotenuse NW and all the angles are given, whence NP = 2608; and PW = 174 *yards*.

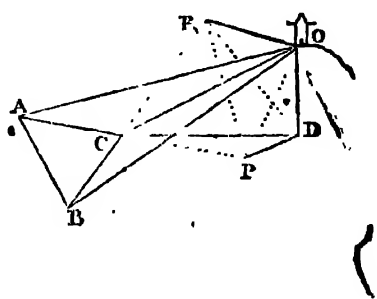
In this example, no reduction is necessary on account of the inclination of the base NS to the horizon.

4. Let BC be a measured base of 370 *yards* on the plane ABC; and suppose marks are set up at the stations A, B, C, and the following angles taken with a sextant to the elevated object O:

At A $\begin{cases} OAC = 20^\circ\ 50' \\ OAB = 80 \quad 18 \end{cases}$

At B $\begin{cases} OBA = 73 \quad 44 \\ OBC = 16 \quad 4 \end{cases}$

At C $\begin{cases} OCB = 149 \quad 10 \\ OCA = 140 \quad 6 \end{cases}$



Required the distance of the object O from the station C, and its height above the plane of the base BC.

The angles of the triangles OAC, OAB, OBC, are

OAC = 20° 50′	OAB = 80° 18′	OBC = 16° 4′
OCA = 140 6	OBA = 73 44	OCB = 149 10
AOC = 19 4.	AOB = 25 58.	BOC = 14 46.

These three triangles form the sides of the pyramid whose vertex is O, and base ACB: we have therefore to find its height OD, and the point D where the perpendicular OD meets the plane of the base.

Calculation.

14° 46′ *sin.* 9.406341
0.593659 } sum 3.161861
BC = 370 log. 2.568202 }
OBC = 16° 4′ *sin.* 9.442096
CO log. 2.603957
OAC = 20° 50′ *ar.co.sin.* 0.118976
AOC = 19° 4′ *sin.* 9.514107
AC = 309 log. 2.567040

9.709730 *sin.* 149° 10′ = OCB
2.871591 log. OB
0.006254 *ar.co.sin.* 80° 18′ = OAB
9.641324 *sin.* 25° 58′ = AOB
2.519169 log. AB = 330.5
0.358676 *ar. co. sin.* 25° 58′, AOB
9.982257 *sin.* 73° 44′, OBA
2.860102 log. AO.

The sides of the triangle ABC { BC = 370, AC = 309, AB = 330.5 } give the angle ACB = 53° 8′ (228).

Let OP, OR, meet AC, BC produced, at right angles in P and R; and suppose OD is the perpendicular on the plane of the base, and join PD, CD, RD. Then OCP = 39° 54′ (the supplement of OCA); and OCR = 30° 50′ (the supplement of OCB);

Then, 39° 54′ *cosine* 9.884889	30° 50′ *cosine* 9.933822
CO log. 2.603957	CO log. 2.603957
CP = 308.2 log. 2.488846	CR = 345 log. 2.537779

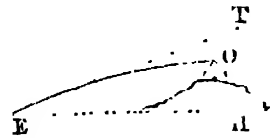
Now in the quadrilateral CRDP (in the plane of the base ABC) we have the sides CP, CR, and their included angle = 53° 8′, whence (226) we get the angle CRP = 57° = CDP (because the angles CPD, CRD being right ones, a circle will circumscribe the quadrilateral), therefore CP and all the angles of the right angled triangle CPD are given; whence the distance CD = 367.5 *yards*; from this side and the hypotenuse CO, the perpendicular OD will be found = 162.3 *yards*.

If the triangle ABC is on level ground, CD is the horizontal distance of the object O from the station C, and OD its height.

OF TERRESTRIAL REFRACTION.

235. As the *apparent* or *observed elevations* of objects are always greater than the *true*, it may not be improper to give a short explanation of Refraction.

Let E be the place of an observer's eye, EH the horizontal line, and O an object, suppose on the summit of a distant hill.

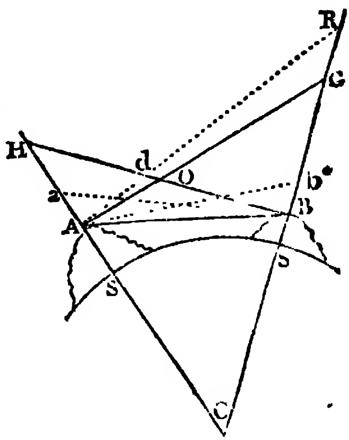


Then if the rays of light proceeded from the object O to the eye at E in a straight line, the object would appear in its true place at O, and OEH would be the elevation (considering EO as a right line); but the rays in passing through the atmosphere are continually attracted or bent downwards from a rectilineal direction, by which means the object is seen in the direction ET, which is supposed to be a tangent to the curve at E, and therefore the apparent or observed elevation is the angle TEH; and the angle TEO, or rather the angle comprehended by TE and a right line from O to E, will be the refraction.

This Refraction which is called the terrestrial, to distinguish it from that which affects the altitudes of the heavenly bodies, is not constant at the same elevation and distance, but is found to vary with the changes in the atmosphere, as heat, a different density, moist vapours, &c. &c. At the distance of 8 or 10 miles it is sometimes no more than about 30 *seconds*, but in particular states of the air we find it amount to upwards of 2 *minutes*.

236. It is a difficult operation to determine the exact quantity of refraction at any particular time. The followir

method however, has been successfully practised in the *Trigonometrical Survey* carried on by order of the Board of Ordnance.



Let A and B be two stations, SS the intercepted or corresponding arc of the earth's circumference, C the centre of the earth; AG, BH, the horizontal lines at A and B drawn to meet CG, CH.

An instrument being at each of the stations A and B, the reciprocal observations are made *at the same instant of time*, which is determined by means of signals or watches previously regulated for that purpose; that is, the observer at A takes the depression (for example) of B while the other person at B observes the depression of A.

If a and b represent the apparent places of the objects A and B, the angle bAB is the refraction at A, and aBA that at B; therefore, half the sum of those angles will be the refraction, if we suppose it equal at each station.

In the quadrilateral AOBC the angles at A and B are right ones, therefore the sum of the other two angles at O and C are equal to two right angles, and consequently the angles OAB, OBA are together equal to the angle C or arc SS, therefore if the sum of the two depressions or angles HBa + GAb is taken from the sum of the angles HBA + GAB or the angle C, the remainder is the sum of both refractions or angles aBA + bAB; therefore *half the difference between the sum of the two depressions and the contained arc* SS (*or angle* C) *is the refraction.*

If one of the objects (B) instead of being depressed, is elevated, suppose to the point R; then the sum of the angles $dAB + dBA$ will be greater than the sum OAB + OBA (or angle C) by the angle of elevation RAG; but if from the sum $dAB + dBA$ we take the depression HBA, there will remain $dAB + aBA$ the sum of the two refractions; therefore, *if the depression be subtracted from the sum of the contained arc and elevation, half the remainder is the refraction in this case.*

It is almost unnecessary to remark that the distance between the places of observation A and B should be known sufficiently near to give the contained arc SS true to a very few seconds of a degree. The refraction however, is generally too minute to be of consequence in the operations with a common Theodolite, which are usually confined to moderate distances.

Of Surveying.

237. Surveying is the Art of laying down the true positions of the principle features, and exhibiting an exact representation of the boundary of a country, or any part thereof, on a plane or paper, so, that the dimensions, &c. may be readily measured by means of a scale of *miles*, *yards*, *chains*, &c. &c. When fields or other inclosures, and Gentlemen's estates are surveyed, not only a correct delineation of the boundaries is required, but the superficial content in *acres*, &c. must be computed. This is called Land Surveying, or Land Measuring.

238. To lay down or make a Map or Plan of any considerable extent of Country, a series of connected triangles should be carried in all directions to its boundaries from a long and well measured base as the foundation: For that purpose the most conspicuous points, as the summits of hills, roofs of

church-towers, &c. &c. must be chosen for stations; and all remarkable objects in view should be intersected at every place where the instrument for taking the angles is set up. When a high pointed spire, or the like, upon which the instrument cannot be conveniently placed, presents itself as a proper situation for carrying on the triangles, it should always be intersected from several stations in order to compare, or correct the connecting distances by a computation from independent triangles.

239. It will be adviseable to observe every angle of the principal triangles if the situations permit; then, as the sum of the three angles of each triangle ought to be very nearly equal to two right ones, the deviations will in some measure, enable us to judge of the accuracy of the work.

240. The sides of the principal triangles should be calculated. But objects situated within those triangles may be laid down by means of a protractor: these objects however, should if possible, be intersected from three stations.

241. The principal triangles and interior objects laid down on a large scale, suppose 5 or 6 inches to the mile, will be a sufficient ground work for Military sketches which are usually drawn by eye without any actual measurement. The method of adapting a scale to the Plan; and enlarging or diminishing it to any particular size is given in *Art.* 167.

242. But the most difficult and tedious operation connected with a Survey, is that of measuring a base-line accurately. We shall therefore recommend a perusal of the Account of the Trigonometrical Survey (236) to those who may engage in an undertaking of this kind when great exactness is required. A base for common surveys may be measured with a 20 *feet deal-rod*: for this purpose a rope not less than 100 *yards* should be stretched very tight along the ground; the rod must then be applied to the rope, and its extremity may be marked with a small pin stuck in the rope to preserve the distance while the rod is removed.

When the measurement is carried on to the extent of the rope, a peg should be driven in the ground and a notch cut on its top exactly under the end of the last rod. The rope must then be taken up and stretched again in the direction of the base, and the measurement continued as before.

When the measurement is carried over hollows or ditches, it may be necessary to support the rod in the middle: it should not however, be made very slender.

If rising grounds intervene, the slant distances must be measured separately as hypothenuses, and afterwards reduced to the corresponding horizontal lines (232): the elevations or depressions may be taken with a Theodolite which has a vertical arc.

It may be necessary to observe, that 20 *feet* should be transferred to the rod from a *standard measure*. And with respect to expansion and contraction, it is pretty well known that well seasoned deal is subject to very little alteration while it is kept dry.

243. If a measurement of this kind be performed with tolerable care, we may safely conclude there will not exist an error of more than $\frac{1}{10}$ of an *inch* in each rod of 20 *feet*, or $26\frac{1}{2}$ *inches* in a *mile*. Supposing however, the accumulated errors amount to 5 *feet* in a base of 2 *miles*, and that a series of triangles whose sides are about 3 *miles* to be determined from such a base, then combining the probable errors from observations made with a Theodolite, the uncertainty in a direct distance of 20 *miles* from the base cannot amount to 30 *yards*. Erroneous as this may be considered, we believe most of the County Maps have been laid down from operations less accurate.

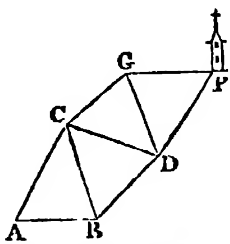
224. If the variation of the Magnetical needle is known, the direction of the meridian may be drawn sufficiently near for a Map or Plan by means of the compass belonging to the Theodolite.

We shall now proceed to such trigonometrical problems as usually occur in the practice of Surveying.

245. Let AB be a base of 2 *miles* or 3520 *yards*; and suppose poles or flag-staffs are set up at the stations A, B, C, D, G; and that the angles at those stations taken with a Theodolite are the following;

namely, CAB =	64° 29′		DCG =	73° 58′
CBA =	75 15		CDG =	41 27
ACB =	40 18		CGD =	54 33
sum	180 2		sum	179 58
BCD =	53 41		DGP =	71 7
CBD =	64 8		GDP =	46 51
BDC =	62 14			
sum	180 3			

It is required to find the distance of the spire P from the station A?



The error in the sum of the three observed angles of the first triangle is 2′; in the second 3′; and in the third 2′. The angle at P in the fourth triangle is supplemental.

But no certain rule can be given for correcting the observed angles: this must be left to the judgment of the observer, who, from circumstances, will seldom be at a loss to point out where the greatest uncertainty lies. To make the calculation however, we will suppose the corrected angles

are CAB =	64° 28′		DCG =	73° 58′
CBA =	75 14		CDG =	41 28
ACB =	40 18		CGD =	54 34
	180 0			180 0
BCD =	53 40		DGP =	71 7
CBD =	64 7		GDP =	46 51
BDC =	62 13		GPD =	62 2
	180 0			

Then (221)

ACB = 40° 18′	.... *sin.*	9·810763	
		0·189237	
AB = 3520	.... log.	3·546543	
CAB = 64° 28′	.... *sin.*	9·955368	
		3·691148	log. CB.
BDC = 62° 13′	*ar. co. sin.*	0·053196	
BCD = 53 40	.... *sin.*	9·906111	
		3·650455	log. BD = 4471 5.
		3 744344	(222.)
CBD = 64° 7′	.... *sin.*	9·954090	
		3·698434	log. CD.
CGD = 54° 34′	*ar. co. sin.*	0·088954	
DCG = 73 58	.... *sin.*	9·982769	
		3 770157	log. GD.
GPD = 62° 2′	*ar. co. sin.*	0·053931	
DGP = 71 7	.... *sin.*	9·975974	
		3·800062	log. DP = 6310·5.

Now from the sides BA, BD, and the included angle 139° 21′ we get the angle BDA = 17° 48′, and AD = 7501·1 *yards*, (226).

And if BDA be taken from 150° 32′ the angle BDP, there remains 132° 44′ the angle ADP, which, with the including sides AD = 7501 1, and DP = 6310·5 will give the distance from P to A = 12659 *yards*.

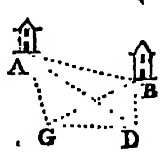
When triangles are carried on from the original base in all directions, the distances towards the extremities may, in some respect, be verified by independent calculations.

N. B. All the principal distances should be laid down from a scale of equal parts, because a triangle can be protracted more accurately with its sides than with the angles.

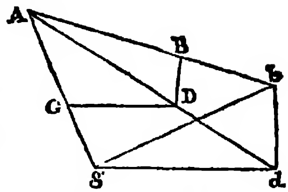
246. Suppose in making a Survey, the distance between the spires A and B has been determined equal to 6594 *yards*; and that G and D are two eminences conveniently situated for extending the triangles.

Now if we observe the angles

at G $\begin{cases} AGB = 65°46' \\ BGD = 23\ 56. \end{cases}$ at D $\begin{cases} ADG = 31°48' \\ ADB = 68\ \ 2. \end{cases}$



It is required to determine the distance GD ?



Construction. At the extremities of any right line *gd* make the angles *bgd* = 23° 56′, A*gb* = 85° 46′; A*dg* = 31° 48′, A*db* = 68° 2′; join the points A, *b*; and make AB (on A*b* produced if necessary) = 6594 from a scale of equal parts; then if BD, DG are drawn parallel to *bd*, *dg*, respectively, GD will be the distance required. For the quadrilaterals A*gdb*, AGDB are similar by construction, and AB in the second figure being = 6594 the distance of the spires, GD must be *that* of the stations on the same scale.

Calculation.

Angles of the triangle A*gd* or AGD.	A*gd* = 109° 42′ A*dg* = 31 48 *g*A*d* = 38 30	of *gbd* or GBD	*gdb* = 99° 50′ *dgb* = 23 56 *gbd* = 56 14.

Now to obtain the angles *gb*A, *d*A*b*, assume *gd* of any length, suppose 1000: then the computation is made exactly as in examp. 9, art. 233.

38° 30′ *ar. comp.* sin.	0·205850	
gd = 1000....... log.	3·000000	
31° 48′........... sin.	9·721774	
	2·927624	log. 846·5, A*g*.

56° 14′ *ar. comp.* sin	0·080238	
gd .. .. log.	3·000000	
99° 50′... .. .... sin.	9·993572	
	3·073810	log 1185·2, *gb*.

The sides *gb*, A*g*, with the included angle A*gb* = 85° 46′ give the

Angles $\begin{cases} gAb = 57° 18′. \\ gbA = 36\ 56. \end{cases}$ Whence *d*A*b* 18° 48′.

Now all the angles in the quadrilateral GABD are given, and the side AB, being = 6594 *yards*, we get AD at one proportion by means of the triangle ADB; then the triangle GAD gives GD = 4694 *yards*, the distance required. Which may serve as a base for determining other distances, or continuing the triangles.

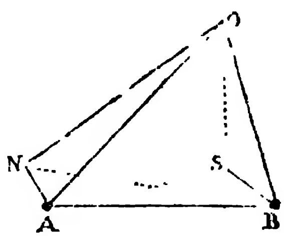
And the method of solution is the same when the stations lie on contrary sides of the given distance AB.

247. When the top of a Church steeple becomes a station in consequence of the wind-vane or a pinnacle having been intersected, the instrument is placed in the most convenient situation, and a reduction of the observed angles will in that case be necessary.

Let A and B represent the wind-vanes on two steeples, their distance having been determined equal to 2587 *yards* = AB; and suppose N is the place of the Theodolite when it is on the steeple A, and S its situation on the steeple B; also, suppose the observed angles at those stations are the following;

$$\text{at N}\begin{cases}\text{ONB} & 45^\circ\,42' \\ \text{ONA} & 96\quad 0\end{cases}\qquad \text{at S}\begin{cases}\text{OSA} = & 70^\circ\,39' \\ \text{OSB} & 147\quad 0.\end{cases}$$

And let the distance from N to the wind-vane A be 11½ *feet*, and that from S to B = 10½ *feet*. Hence it is required to find the angles OAB, OBA, or what the observed angles to the distant object O would be if the instrument was at the points A and B?



The angles 45° 42′, 70° 39, and 63° 39′ their supplement, with the distance AB = 2587, will give 2066 and 2724 *yards*, the distances BO, AO, nearly.

Then (224) as AO : *sin.* 96° (ONA) :: NA, 3·83 *yards* : *sin.* 5′ nearly, the angle AON.

And 96° — 45° 42′ = 50° 18′, the angle BNA;

Hence, AB : *sin.* 50° 18′ :: NA : *sin.* 4′ nearly the angle ABN.

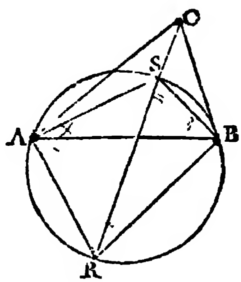
Therefore the sum of the two angles NOB, NBO is *greater* than the sum of the two angles AOB, ABO by the difference of AON, ABN; consequently ONB is *less* than OAB by 1′; therefore AOB is = 45° 43′.

Again, as BO : *sin.* 147° (OSB) :: 3½ *yards* (SB) : *sine* 3′ nearly, the angle SOB. And, AB : *sin.* 142° 21′ (ASB) :: 3½ (SB : *sin.* 3′ nearly, the angle SAB.

Now the angles SAO, SOA together are *less* than both the angles BAO, BOA by the sum of the angles SAB, SOB; therefore ASO is *greater* than ABO by that sum; hence the angle ABO = 70° 39′ — 6′ = 70° 33′. And BO, AO calculated with the corrected angles 45° 43′ and 70° 33′, are 2065·3 and 2720·2 *yards*.

It is not necessary that the angles ONA, OSB should be very accurately taken; but the distances NA, SB must be carefully measured.

248. If A, B, C, be three objects whose distances from each other are AB = 4516, AC = 4809, BC = 3018 *yards*; and suppose at the station S we observe the angles CSB = 117° 56′, BSA = 110° 12′; it is required to find the distances from the station to the three objects.



Construction. If the triangle ABC be laid down with the three given distances, and segments of circles described upon any two sides to contain the angles they subtend (172), the intersection of the arcs will evidently be the station, whether it falls within, or without the triangle. But the following method is rather more simple — About AB describe a circle so that the segment ABS shall contain the angle 110° 12′; make the angle BAR = 62° 4′ the supplement of 117° 56′ (CSB), join CR; and S, where it intersects the circle, is the station. For if AS, SB, BR are drawn, the angle ASB is = 110° 12′ by construction; and RSB being equal to RAB (70) or 62° 4′, the angle CSB which is its supplement, will be 117° 56′ the other observed angle.

Calculation. The three sides 4516, 4809, 3018 give the angle ABC = 76° 28′ (22).

Angle ABR (= ASR the supplement of ASC) = 48° 8′
BAR .... = 62 4
ARB .... = 69 48, these with the side AB give BR = 1251·3.

The angle RBC = 48° 8′ + 76° 28′ = 124° 36′ which, with the two including sides, give RCB = 32° 47′, and CRB = 22° 37′.

Now SAB = SRB = 22° 37′; therefore all the angles of the triangles ASB, BSC are given;

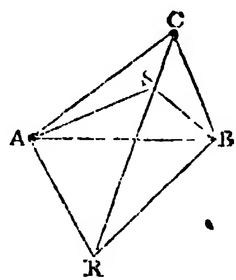
namely, SAB =	22°37′	SCB =	32°47′
ASB =	110 12	CSB =	117 56
SBA =	47 11	SBC =	29 17

Whence (221), the distances SA, SB, SC, are found to be 3530, 1851, 1672 *yards*, respectively.

When the station is without the triangle (suppose at R) it is evident the circle must be described so that the outward segment ARB shall contain the whole observed angle ARB; then if the angles ABS, BAS be made respectively equal to the observed angles ARC, BRC, and CR drawn through S, R will be the station.

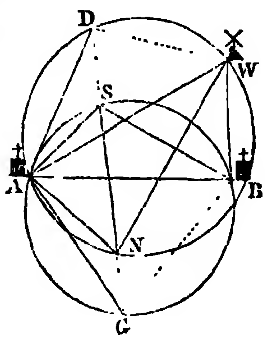
If the whole observed angle ARB should be equal to the supplement of the angle ACB, the circle will pass through the point C; in which case the problem is indeterminate: for the angles standing on the chords BC, AC would be the same in all points of the arc ARC, (70.)

249. The last problem will be found useful in reconnoitring a country with a map or plan; for the angles taken to any three objects which are laid down, will determine the situation of the observer. A small pocket sextant is the most convenient instrument for measuring the angles. And it appears from the preceding construction that it is not necessary to describe a circle. For example, if the station be within the triangle, then the angles BAR, ABR being made equal to the supplements of the observed angles BSC, ASC, the intersection of AR and BR gives the point R; then if the angle ABS be made equal to the angle ARC, BS will meet RC in S the station. On the contrary, when the place of observation is without the triangle, the angles ABS, BAS, are made equal to the observed angles ARC, BRC, respectively, then CR being drawn through S, and the angles ABR, BAR made equal to ASR, BSR, BR and AR will meet CR in R the station.



In this latter case however, when the point S falls near the object C, the *construction* may give the point R considerably wide of the truth.

250. In making a Survey we found two spots N and S conveniently situated for stations; and at S took the angles NSA = 52° 58′, NSB = 55° 4′, to the spires A and B: But at N an intervening height hid the spire B; we therefore observed the angle between the wind-mill W and station S, and found it = 38° 4′, and then took the angle SNA which was 41° 46′. Now AW, AB, BW being respectively equal to 5232, 4490, 2678 *yards*, it is required to find the distance SN?



Construction. With the three given sides lay down the triangle AWB. Then about AB and AW describe circles so that the segment ASB shall contain an angle of 108° 2′ (52° 58′ + 55° 4′); and the segment ANW an angle of 79° 50′ (38° 4′ + 41° 46′). Draw the chord AD to subtend an angle (AND) = 41° 46′, and the chord AG to subtend an angle (ASG) = 52° 58′; join DG; and the intersections S, N, will be the stations. For if SB, SA; NA, NW are drawn, the angles at S and N to the three objects will be equal to the observed angles, by the construction, and *Art.* 70.

Calculation. Draw DW, GB. Then all the angles of the triangles ADW, AGB are given;

viz. DNW = DAW = 38° 4′		GSB = GAB = 55° 4′
DNA = DWA = 41 46		GSA = GBA = 52 58
ADW = 100 10		AGB = 71 58

As *sin.* ADW : 5232 (AW) :: *sin.* DWA : 3540·6 = AD.
And *sin.* AGB : 4490 (AB) :: *sin.* GBA : 3769 5 = AG.

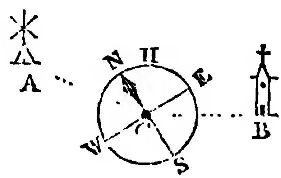
The sides of the triangle AWB give the angle WAB = 30° 47′

DAW = DNW = 38° 4′
WAB .... = 30 47
BAG = BSG = [illegible] 4
DAG = [illegible] with this angle and the including sides we get ADG = 29° = AWN; therefore in the triangle AWN all the angles and the side AW are given, whence AN = 2577; then, as the angles of the triangle ASN are also given, we get SN = 3217 *yards*.

And the method of construction and calculation will vary little from the preceding, howsoever posited the stations may be in respect of the three given objects.

Of Surveying with the Compass.

251. In this operation we do not measure the angles subtended by distant objects in the same manner as with a Theodolite, but take their angular distances or bearings from the *magnetical meridian*. Thus if NS represents the magnetic needle or meridian, W the west, and E the east; and suppose the sights on the Compass are directed to the wind-mill A: then if the angle ACN is 40°, for example, the wind-mill is said to bear NW 40°, or 40° westward from N the magnetical north. Or if the sights are directed to the spire B, and the angle SCB is 64° then the spire bears SE 64°.

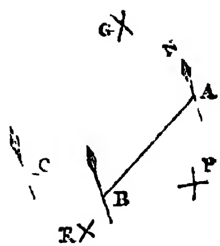


If CH represents the direction of the *true meridian*, the angle NCH is called the *variation* of the magnetical needle; which, at this time, is about 23° or 24° westward at London.

252. Let A and B be two stations bearing SW 61° and NE 61° from each other;

and suppose at A the objects	G bears	NW	29°
	P	SW	18
	R	SW	54

and at B $\left\{ \begin{array}{lll} \text{G bears} & \text{NE } 22 \\ \text{P} & \text{SE } 83 \end{array} \right.$



From A draw the NW 29° and SW 18° lines; and from B the NE 22° and SE 83° lines; then their intersections will give the places of the objects G and P.

Suppose C to be a third station,

where the objects $\left\{ \begin{array}{lll} \text{G bears} & \text{NE } 51° \\ \text{P} & \text{SE } 70 \\ \text{R} & \text{SE } 31 \end{array} \right.$

Then from G and P draw two lines parallel to NE 51°, and SE 70°, and their intersection will determine the station C.

And the intersection of the SW 54° line from A, with that of the SE 31° line from C gives the position of the object R.

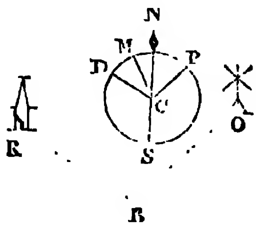
253. Since the magnetical meridians are considered as parallels, it is evident that the bearings of any two objects already laid down will give the place of the observer; but every intersection should be as near a right angle as circumstances will admit. The bearings of all conspicuous objects however, ought to be taken at every station, by which means a great number may be fixed from several intersections.

This is a very expeditious method of laying down the relative situations of the prominent points of a small tract of country. The compasses most convenient are about 3½ *inches* in diameter; and may be carried in the pocket. They are easily fitted to the top of a stick or staff which must be stuck upright in the ground that the needle may play freely. These compasses are divided into *degrees* only, and consequently much accuracy cannot be expected in Surveys of this kind: they

may serve however, as the ground work of, or for correcting Military sketches.

For temporary use, it will sometimes be necessary to measure a distance by *pacing*, in order to adapt a scale to the plan or sketch (167).

254. The Compass will also be found useful in reconnoitring a country with a map or plan when the direction of the meridian is laid down, and we know the magnetical variation. Let SN be the direction of the true meridian on a map; and suppose the wind-mill O bears NE 68°, and the spire R, NW 36° by the compass; also let the variation be 23° W.



Make the angle MCN = 23°, then CM will represent the magnetical meridian: let the angle MCP = 68°, and MCD = 36°; then if OB, RB are drawn parallel to PC, DC, respectively, the intersection B will be the place of observation on the map or plan. If however, the intersection (B) is very acute or obtuse, the position thus determined may be considerably wide of the truth.

ADDITIONAL EXAMPLES

IN

PRACTICAL GEOMETRY, TRIGONOMETRY, and MENSURATION.

1. If the diagonal of a square redoubt be 67 yards; what is the length of the side?

Ans. 47·376 &c. *yards.*

2. The sides of three squares being 4, 5, and 6 feet; then how long is the side of that square which is equal to all three?

Ans. 8·7749 *feet*, nearly.

3. If the lengths of two lines are 20 and 30 inches; what is the length of that line which is a geometrical mean between them?

Ans. 24·4949 *in.* nearly.

4. If the diameter of a circle be 50 yards; what is the length of a chord which is 5 yards distant from the centre?

Ans. 48·9899 *yds.*

5. If a point be 20 inches distant from a circle whose diameter is 20 inches, and a line 30 inches long be drawn from that point to the circumference; what is the length of that part of the line which is without the circle?

Ans. $26\frac{2}{3}$ *inches.*

6. Suppose in the last example, the line is drawn from the given point to make the intercepted chord 10 inches; what is the length of the part without the circle?

Ans. 23·7228 &c. *inches.*

7. In the preceding example, what is the length of the tangent to the circle drawn from the given point?

Ans. 28·284 &c. *in.*

8. To what extent on the surface of the sea (exclusive of the effect of refraction) can a person see from the top-mast-head of a man of war, his height above the water being 30 *yards*, and the earth's diameter 7960 *miles*?

Ans. 11·6 *miles*, nearly.

9. If a line 10 inches long be cut according to mean and extreme proportion; what are the lengths of the two parts?

Ans. 6·18 and 3·82 *in.* nearly.

10. If the base of a triangle be 40, and the other two sides 30 and 20; what is the length of its perpendicular?

Ans. 14·52 &c.

11. If the base of a triangle be 40, and the two sides 30 and 20; what are the segments of the base made by a line bisecting the vertical angle?

Ans. 24 and 16.

12. If the diameter of a circle be 30; what is the side of the inscribed equilateral triangle?

Ans. 25·98 nearly.

13. If the side of an equilateral triangle be 10; what are the radii of the inscribed, and circumscribing circles?

Ans. 2·8868 and 5·7736 nearly.

14. The side of a square being 10; then what is the radius of its circumscribing circle?

Ans. 7·071 &c.

15. If the side of a regular pentagon be 10; what are the radii of its inscribed, and circumscribing circles?

Ans. 6·882 and 8·506 nearly.

16. If the radius of a circle be 10; what are the sides of the regular inscribed trigon, tetragon, pentagon, hexagon, octagon, and decagon?

Ans. 17·32—14·142—11·756—10—7·654—6·18, nearly.

17. A plan of a fortified town has a scale of 100 toises which is 1·6 inches in length; the plan is 30 inches long, and 24 broad; now what will be the size when it is copied to a scale of 6 inches the English mile?

Ans. 13·6 *in.* long, and 10·9 broad.

18. If the length of a pair of proportional compasses be 7 inches; how far from the ends is the centre answering to the division 5 on the line of Lines?

Ans. $1\frac{1}{6}$ and $5\frac{5}{6}$ *inches.*

19. Suppose the length of a pair of proportional compasses to be exactly 9 inches; how far from the ends must the centres be for enlarging or diminishing a plane surface twice, and a solid three times?

Ans. 3·728 and 5·272 *in.* in the *former case.*
3·685 and 5·315 *in.* in the *latter.*

20. If the length of a cannon be 8 *f.* 10 *in.* its diameter at the breech $19\frac{1}{8}$ *in.* at the mouth $14\frac{1}{6}$ *in.* at what distance would the outer surface meet the axis of the bore supposing both were produced?

Ans. $25\frac{5}{21}$ *feet*, from the muzzle.

21. How many degrees, &c. are contained in that arc of a circle whose length is equal to the radius?

Ans. 57°·295779 nearly.

22. If the line of numbers from 1 to 10 on a *logarithmic* or *Gunter's Scale* is a foot; required the distance from 1 to 5.— And what is the distance from 10 on the line of numbers to 40° on the *line of tangents*?

Ans. 8·3876 &c. and 0·914 &c. *inches.*

23. The length of a line of chords of 90° being $4\frac{1}{4}$ inches; then what is the length of 45° on the same line?

Ans. 2·3 *in.* nearly.

24. If the radius of a circle be 20; what are the lengths of the *sine*, *cosine*, *tangent*, *cotangent*, *secant*, and *cosecant* of 30°?

Ans. 10—17·32—11·547—34·641—23·094—40.

25. If the base of a right-angled triangle be 4, and the perpendicular 3: what are the lengths of the *sine*, *cosine*, *tangent*, and *cotangent* of the least angle, if the radius be 1?

Ans. 0·6 — 0·8 — 0·75 — 1·333 &c.

26. If the base of a right-angled triangle be 0·28, and the adjacent acute angle 59° 11′; what are the other sides?

Ans. 0·5466, and 0·4694.

27. The base of a right-angled triangle being 74·7 *yards*, and its opposite angle 21° 13′; what are the other sides?

Ans. 192·4, and 206·4 *yds.*

28. The hypotenuse of a right-angled triangle being 5472 *feet*, and one of the acute angles 29° 51′; then what are the other sides?

Ans. 4746 and 2723·5 *feet.*

29. If the three angles of a plane triangle are 106° 41′, 46° 24′, and 26° 55′, and the side opposite the greatest angle = 297 6 *yds.* then what are the other sides?

Ans. 225, and 140·7 *yards.*

30. Suppose the angles of a plane triangle to be as in the preceding example, and the side opposite the least angle 297·6 *feet;* required the other sides?

Ans. 476·1 and 629·7 *feet.*

1. The hypotenuse of a right-angled triangle being

14 *f.* 10 *in.* and the base 10 *f.* 7 *in.* then what is the perpendicular?

Ans. 10 *f.* 4·7 *in.*

32. Two sides of a triangle being 311 and 397 yards, and the angle opposite the greater of those sides $=$ 38° 33′; then what is the third side?

Ans. 589·7 *yds.*

33. Suppose two sides of a triangle are 311 and 221 yards, and the angle opposite the least of those sides is 38° 33′; required the third side?

Ans. 349·4, or 137·04 *yds.*

34. If two sides of a triangle are 179·8 and 121·6 feet, and the included angle 79° 51′; what is the third side?

Ans. 198·5 *feet.*

35. The base and perpendicular of a right-angled triangle being 1139, and 1074 yards; required the acute angles, and hypotenuse?

Ans. 43° 19′ — 46° 41′ — hypot. $=$ 1565·5 *yds.*

36. If an angle of a triangle be 129° 34′; and the ratio of the including sides as 4 to 7; what are the other two angles?

Ans. 32° 32′ 7″ — 17° 53′ 53″.

37. How many inches subtend an angle of 1″ at the distance of 7 miles?

Ans. 2·1 nearly.

38. Suppose the sides of a triangle are 11272, 13141, and 11799 yards; required the angles?

Ans. 69° 34′½ — 59° 38′½ — 50° 47′.

39. If the sides of a triangle have the proportion of $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$; what are the angles?

Ans. 117° 16′ 46″ — 36° 20′ 10″ — 26° 23′ 4″.

40. Let the base of a right-angled triangle be 30, and the ratio of the other two sides as 1 to 2; what are those sides?

Ans. 17·32, and 34·64 nearly.

41. If the hypotenuse of a right-angled triangle be 100, and the other sides as 1 to 2; what are those sides?

Ans. 44·72 and 89·44 nearly.

42. If the hypotenuse of a right-angled triangle be 40, and the sum of the other two sides 50; what are those sides?

Ans. 11 771 — 38·229 nearly.

43. Suppose the hypotenuse of a right-angled triangle to be 40, and the difference of the other sides 10; required the sides?

Ans. 22 839 — 32·839 nearly.

44. If the base of a right-angled triangle be 40, and the sum of the other sides 80; what is the perpendicular?

Ans. 30.

45. If the perpendicular of a right-angled triangle be 40, and the difference of the other sides 10; what are those sides?

Ans. 75 and 85.

46. Suppose a regular pentagon whose side is 170 fathoms, to be fortified; and that the salient angle of the bastion is 71°, and its face 47 fathoms; required the flank, and curtain, supposing the line of defence is perpendicular to the flank?

Ans. Flank 25·65
Curtain 64·57.

47. If a square whose side is 170 fathoms is regularly fortified, and the salient angle of the bastion 61°; what are the principal dimensions if the length of the face of the bastion, is to that of the flank, as 7 to 3; the line of defence being perpendicular to the flank?

Ans. Face of bastion 46·6—Flank 20—Curtain 69·8.

48. If at the top of a mountain the true depression of the horizon of the sea is found to be 1° 31′; what is the mountain's height, supposing the earth a sphere whose diameter is 8000 miles?

Ans. 1·4 *miles,* nearly.

49. In surveying with a compass an object bore NE 50°; and when we had gone 170 paces in a SE 55° direction, its bearing was NE 6°. Required its distance from each station?

Ans. 214, and 237 *paces.*

50. Wanting to know the breadth of a river, we measured a straight base of 30 chains along the bank, and at its extremities took the horizontal angles 64° 11′, and 78° 38′ to an object on the opposite shore. Hence the breath is required?

Ans. 964 *yards.*

51. From the top of a hill I observed two mile stones in the same direction on level ground; the depression of the nearest was 14° 3′; and that of the other 3° 56′ below the horizontal line: hence the height of the hill is required?

Ans. 501 *feet.*

52. Having observed the elevation, of an object on the top of a distant hill, and found it 2° 27′, we measured a base of 520 yards on sloping ground directly towards the object, and at that end the object was elevated 3° 4′. Now the farthest extremity of the base was found to be 10 *feet,* higher than the other. Hence the height, and distance of the hill are required?

Ans. Height above the lowest end of the base 127 *yds.*
Distance from that end 2371 *yds.*

53. To find the height, and distance of an object on the top of a hill, we measured a base of 470 yards on sloping

ground which was inclined to the horizon in an angle of 4° 44′; and then observed the horizontal angles between the base and object at the lower and upper ends of base, and found them to be 91° 12′, and 72° 57′, respectively; also at the lower end of the base, the object was elevated 4° 3′. Hence the height and distance of the hill are required?

Ans. Horizontal dist. from the lower end of the base 1640 *yds.*
Height above that end 116 *yds.*

54. At the top and bottom of a tower which stood on a hill near the sea shore, we observed the depressions of a ship at anchor to be 1° 39′, and 1° 9′, respectively: hence the height of the hill, and also its distance from the vessel are required; the tower itself being 72 feet high?

Ans. Bottom of tower above the sea 166 *feet.*
Horizontal distance of ship 8246.

55. To obtain the height, and distance of an object on the summit of a hill I measured a base of 450 *yards* on level ground, and set up marks at its extremities equal to the height of the eye. At one end of the base the angle between the other end and the object was found with a *sextant* to be 74° 35′; and at the other end 77° 41′ where the elevation of the object was observed = 6° 29′. Hence the height of the hill, and its distance from each extremity o the base are required?

Ans. Height of the hill 105·3 *yds.*
Distances 926·2.
938·8.

56. In surveying with a *compass*, a spire bore NE 18°, distant 2 miles; and the bearing of a wind-mill was NW, 20° now the distance of the wind-mill from the spire was known to be 1¼ miles: hence its distance from the station is required?

Ans. 2395, or 3153 *yards.*

57. A ladder 28 feet long will reach from one side of a ditch which is 20 feet broad, to the top of a wall on the other side: what is the height of the wall?

Ans. 19·6 *feet.*

58. From the top of a work 15 feet high, a point-blank shot struck an object on the ground at the horizontal distance of 120 yards. What was the depression of the piece?

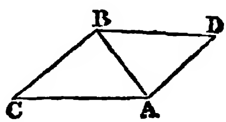
Ans. 2° 23′.

59. Two forts commanding the mouth of a harbour bore SE 16°, and SW 24°½, distant 1¾ and 2½ miles, respectively: required the distance from one to the other, and also their bearing?

Ans. Distance 2870 *yds.*
Bearing 68° 41′ NE and SW.

60. At the extremities of the base AB of 40 *chains*, we took the following angles with a theodolite to the elevated objects C and D:

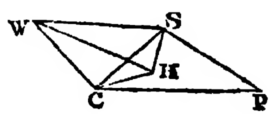
At A $\begin{cases} CAB = 51° 6' \\ DAB = 83\ 5 \\ C\ \textit{elevated}\ 4°\ 17' \\ D\ \textit{elevated}\ 3\ \ 8 \end{cases}$ At B $\begin{cases} CBA = 90° 56' \\ DBA = 48\ \ 3 \end{cases}$



Hence the distance from C to D; and also their heights are required?

Ans. Dist ...... 2129 *yds.*
Height of C 107·1
of D 47·6

61. Let W be West Wycombe church, H High Wycombe church, and P Penn beacon-pole: Now at the stations C and S we took the following angles with a theodolite.



viz. at C $\begin{cases} WCS = 108° \ 14' \\ SCH = 28 \ \ 20 \\ SCP = 33 \ \ 51. \end{cases}$ at S $\begin{cases} WSC = 42° \ 42' \\ CSH = 25 \ \ 26 \\ CSP = 126 \ \ 20. \end{cases}$

By a previous operation the distance WH (between the churches) was found to be 4646 *yards*. Hence the distance from Penn beacon to West Wycombe church is required?

Ans. 9144 *yds.*

62. In reconnoitring a county by the help of a map, we perceived two spires A and B in the same direction, A being the nearest; we then observed the angle subtended by A and a third spire C and found it 41° 52′: now the distance of A and B, measured on the scale to the map, was 3640 *yards*, of A and C 4280, and of B and C 5460. Required the distances to the spires A and C?

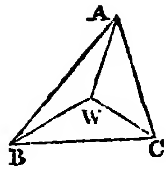
Ans. From A 4527 *yds.*
From C 6403.

63. In surveying with a *pocket-sextant* I observed the angle subtended by two churches A and B = 45° 30′, and that between A and another church C = 25° 40′, all in the horizontal plane nearly: The distance from A to B was $2\frac{1}{4}$, from A to C $2\frac{1}{2}$, and from B to C $4\frac{1}{4}$ miles, the church A being the nearest. Hence the place of observation is required?

Ans. 3314 *yards* from A.
5500 .... from B.
7146 .... from C.

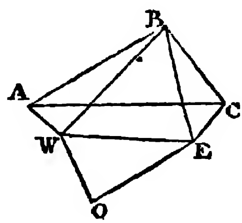
64. From the top of the tower A we observed the angle BAW between the wind-mill W and the spire B, and found

it 23°; but at W the tower A could not be seen from the ground, we therefore took the angle BWC subtended by the spires B and C, which was 123½°. Now the three distances AB, AC, BC were known to be 5450, 4600, 4850 *yards*, respectively. Hence the situation of the wind-mill is required?



Ans. Wind-mill from A 3130 *yds*.
from B 2845.
from C 2659.

65. The distance (WE) of the stations W, E, and also the situation of the object O became necessary in carrying on a survey: now A, B, and C, were three known objects, the distances being AC = 4060, AB = 3200, and CB = 1840 *yards*; but at the station W the object C could not be seen; and an intervening height hid the object A at the other station E; we therefore set up marks at W and E and took the following angles:



namely,

At W $\begin{cases} AWB = 96° \ 10' \\ BWE = 48 \ \ 30 \\ OWE = 58 \ \ 44 \end{cases}$ At E $\begin{cases} BEC = 50° \ \ 4' \\ BEW = 70 \ \ 56 \\ WEO = 32 \ \ 50. \end{cases}$

Hence WE, EO, and WO are required?

Ans. WE = 2697 *yds*.
EO = 2306
WO = 1463

66. In walking along a straight road directly west, I observed two spires A and B both bearing NE 22½°, the nearest being A; an hour afterwards a third spire C and the spire B appeared in one direction; and the next hour brought C and A in a right line; the distance of A from B (on a

map) was $1\frac{3}{4}$ miles, of A from C 2 miles, and that of B from C $3\frac{1}{4}$ miles. How far did I walk *per* hour, supposing the rate equable?

Ans. 7867 *yds.* the whole distance walked, or $3933\frac{1}{2}$ *per* hour.

67. The base of a parallelogram being 61, and its perpendicular $37\frac{1}{2}$ *feet*; what is the content in yards square?

Ans. $254\frac{1}{6}$.

68. The length and breadth of a rectangular field are 13 *chains*, 64 *links*, and 11 *ch.* 9 *lin.* Required the content in *acres*?

Ans. 15·12676.

69. The parallel sides of a trapezoid are 37 *f.* 10 *in.* and 16 *f.* 8 *in.* and their perpendicular distance 11 *f.* 6 *in.* What is the area?

Ans. $313\frac{3}{8}$ *feet.*

70. If the base of a triangle be $17\frac{1}{9}$ yards, and its perpendicular $11\frac{1}{7}$ yards; what is the area in *feet*?

Ans. $862\frac{16}{55}$.

71. If the side of a rhombus is $29\frac{1}{2}$ feet, and the acute angle 62°; what is the content in *yards*?

Ans. 85·38 nearly.

72. The sides of a triangular field being 171, 161, and 145 yards; then what is the area in *acres*?

Ans. 2·2527 nearly.

73. The sides of a quadrangular field being successively 26, 20, 16, and 10 poles, and the angle (taken with a theodolite) included by the two longest sides = 56°. Required its content?

Ans. 287·676 *poles*, or 1 *ac.* 127·676 *pol.*

74. The breadth of a ditch at top being 72, at bottom $38\frac{3}{4}$,

the sloping sides $26\frac{2}{3}$ and 20 feet, and the top and bottom of the ditch horizontal. Required the area of the perpendicular section?

Ans. $885\frac{1}{3}$ *feet.*

75. The area of the perpendicular section of a ditch being 135 feet, the breadth at top 30, and at bottom 15 feet. What is the depth?

Ans. 6 *feet.*

76. If the area of the perpendicular section of a ditch be 154 feet, its depth $5\frac{1}{2}$ feet, and the breadth at top, to that of the bottom, as 9 to 5: what are those breadths?

Ans. 36 and 20 *feet.*

77. The area of a right-angled triangle being 605, and the ratio of the base to the perpendicular as 2 to 5: what are those sides?

Ans. 22 and 55.

78. What is the side of that equilateral triangle whose area is 100?

Ans. 15·197 nearly.

79. If the side of an equilateral triangle be 10; what will be the side of another equilateral triangle whose area is *one-fourth* of the former?

Ans. 5.

80. If the area of a triangle is 1000, and the sides are in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$; what are those sides?

$$\textit{Ans.}\ \left.\begin{array}{r} 40\cdot074 \\ 50\cdot093 \\ 66\cdot791 \end{array}\right\}\ \text{nearly.}$$

81. If the hypotenuse of a right-angled triangle be 17, and its area 60; what are the base and perpendicular?

Ans. 8 and 15.

82. How many acres would be contained within the boundaries of the pentangular fortification, *Examp.* 46, supposing it completed?

Ans. 126089 *yds.* or 26·051, &c. *acres.*

83. If the equal sides of an isosceles triangle are each 17, and its area 120; what is the base?

Ans. 16.

84. If the diameters of two concentric circles are 20 and 30; what is the content of the *annulus* or space contained by the circumferences?

Ans. 392·7.

85. If the area of a circle be 100; what is the area of its inscribed square?

Ans. 63·66 nearly.

86. If the base and perpendicular of a right-angled triangle are each 1; what is the area of a circle having the hypotenuse for its diameter?

Ans. 1·5708 nearly.

87. If the circumference of a circle be 1000; what is its area?

Ans. 59577.

88. If the area of the sector of a circle be 100, and the length of its arc 20; what is the angle of the sector?

Ans. 114° 35·5′ nearly.

89. If the centre of a circle whose diameter is 20, is in the circumference of another circle whose diameter is 40; what are the areas of the three included spaces?

Ans. 173·852.
140·308.
1116·332.

90. How many square feet of board are required to make a rectangular box whose length shall be $3\frac{1}{2}$ *feet*, breadth 2 *feet*, and depth 20 *inches*?

Ans. $32\frac{1}{3}$.

91. What quantity of canvas is necessary for a conical tent whose height is 8 *feet*, and the diameter at bottom 13 *feet*?

Ans. $210\frac{1}{2}$ *feet* square.

92. What would a circular reservoir whose diameter at top is 40 *yards*, at bottom $38\frac{2}{3}$ *yards*, and the side or slant depth 11 *feet*, cost the lining with brick-work at 3*s*. 10*d*. the square yard?

Ans. 311*l*. 18*s*. 2*d*.

93. The inside of an hemispherical dome cost 100*l*. the gilding at 8*d*. the foot; what was its diameter?

Ans. 43·7 *feet*.

94. If the diameter of a globe be 8 inches; what is the diameter of another globe three times as big?

Ans. 11·538 *in.* nearly.

95. If the area of the perpendicular section of a rivulet is $4\frac{1}{2}$ feet, and the velocity of the water 30 feet per minute; how much would it supply in 24 hours?

Ans. 1451213 *gall.* wine measure.

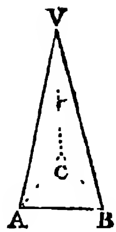
96. Suppose a sack when laid flat is 2 feet broad, and 5 feet long; how many gallons, dry-measure, will it contain if it has a circular bottom, and 9 inches is left for tying the top?

Ans. 34·8 *gall.* nearly.

97. The outer and inner circumferences of the ring of an anchor being respectively 50 and 25 inches; what is its weight, supposing 3·61648 cubic inches of iron weigh a pound avoirdupois?

Ans. 129 *lb*.

98. Suppose the triangle BCA is the base of a pyramid, V its vertex, the side BC = 30 *feet*, 10 *inches*, and the angles



VAC = 20° 50′	VAB = 80° 18	VBC = 16° 4′
VCA = 140 6	VBA = 73 44	VCB = 149 10

What is the cubic content?

Ans. 1709·8565 *cubic feet*, nearly.

99. If a cask which is two equal conic frustums joined together at the bases, has its bung diameter 34, head diameter 27, and depth 50 inches; how many gallons, ale measure, will it contain?

Ans. 130, nearly.

100. What is the difference between a bushel, *running measure*, when measured with a Winchester-bushel which is $18\frac{1}{2}$ inches in diameter, and measured with another bushel only 12 inches in diameter, supposing the *cop* or *cap* or conical part is $\frac{1}{4}$ of the diameter in height?

Answer. The buyer loses 301 cubic inches, or upwards of a gallon in every bushel by the narrowest measure.

101. If a piece of squared timber be 25 feet long, the side of the greater end 20 inches, and that of the less 16; what length must be cut off the less end to make 10 cubic feet?

Ans. 5 *f.* 4 *in.*

102. If the depth of a vessel in the form of a conic frustum, be 16 inches, and the top and bottom diameters in the proportion 5 to 3; what are those diameters, supposing the vessel holds 20 wine gallons?

Ans. 23·722, and 14·233 inches.

103. Suppose the following are the dimensions of the bed of a waggon,

viz.	length	7 *feet.*
	depth	2
	breadth at top behind	5
	——— at bottom	$4\frac{1}{3}$
	breadth at top in front	$4\frac{2}{3}$
	——— at bottom	4

How many bushels, dry measure, will it contain?

Ans. 405 *gall.* or 50 *bush.* 5 *gall.*

104. If the salient angle of a bastion be 71°, and each of its faces 50 fathoms: required the number of cubic yards in that part of the rampart next the faces, supposing AORS *Art.* 265, *Examp.* 2, is the profile or section perpendicular to the face at the angle of the shoulder?

Ans. 13477 *yds.*

105. Suppose the breadth of a circular ditch at top is 36, at bottom $19\frac{1}{3}$, the outer slope 10, and inner slope $13\frac{1}{3}$ feet, respectively; required its capacity in cubic yards; the diameter of the inner circle or edge of the ditch being 600 feet, and the top and bottom of the ditch horizontal?

Ans. 16433 *yds.*

END OF THE FIRST VOLUME.

Printed by W. Glendinning,
25, Hatton Garden.

Errata in Vol. I.

Page	Line	*for*	*read*
63	19	$\frac{24}{9}$	$\frac{42}{9}$
107	10	·03802	·03082
112	23	4	·4
118	9	$\frac{36}{5}$	$\frac{16}{5}$
185	14	*ae*	*are*
233	5	but RO^2	but RC^2
237	16	with same	with the same
244	24	*castramentation*	*castrametation*
246	6 from bott.	ORC	ORA

In the Logarithms.

Log. of 6241 *for* 4251 *r.* 5254

of 6181 *for* 0612 *r.* 1612

Log. tang. 18° [illegible]' *for* 9·3 &c. *r.* 9·5 &c.

cosine 22° 15′ *for* 969 &c. *r.* 966 &c.

cotang. 45° *for* 1· *r.* 10·

Errata in Vol. II.

Page	Line	*for*	*read.*
7	13	$d^2 + 8d$	$c^2 + 8c$
64 examp. 3.		$x^2 - 1$	$x^2 + 1$
75	1	$l(a\sqrt{c})^{\frac{1}{2}}$	$l(a - \sqrt{c})^{\frac{1}{2}}$
77	3	$(a -)^{\frac{1}{3}}$	$(a - x)^{\frac{1}{3}}$
79	2	$-\frac{1}{36}\sqrt{6}$	$= \frac{1}{36}\sqrt{6}$
	12	$\times z^2$	$+ z^2$
98	20	$\frac{1}{2}a^2$	$\frac{1}{4}a^2$
100	20	$2 - 3$	$z - 3$
203	20	$-\frac{8}{5}a^2b^3 + \frac{3}{10}ab^4$	$+\frac{8}{5}a^2b^3 + \frac{3}{10}ab^4 - b^5$
204	1	$+ a^3b^2$	$+ 50a^3b^2$
	14	$\frac{2}{3}xy$	$\frac{3}{4}xy$
	18	$\frac{7}{12}xy$	$\frac{9}{12}xy$
205	4	$9x$	$9x^2$
206	5	$10a$	$15a$
207	4	$5ax$	$5ax^2$
209	14	$\frac{z}{x} + \frac{z^2}{4^2}$	$z + \frac{z^2}{x^2}$ in the *Ans.*
	25	$-\frac{1}{2}xy.$	$+\frac{1}{2}xy$
237	12	$-$ AD	$=$ AD
270	1	*Every &c*	*Every circumscribing parallelogram having its sides parallel to two conjugate diameters equal, &c*
347 bott. line		of $\frac{1}{4}m^2$ &c.	of $\frac{1}{4}m^2$, or m is the least possible, &c.

CORRECTIONS.

Vol. I. 2d. Edit.

Page.	*Line.*	*for*	*read.*
39	17	01·9	·042
121	15	18	10
126	4	15	45
131	23	$\frac{29}{48}$	$\frac{29}{48}$ in some copies.
141 *Ex.* 23.		23	25.

Vol. II.

120	26	$-\frac{1}{2}$	$-\frac{1}{2}d$
146	25	$b+^{3-2}b^2$	$b+a^{3-2}b^2$ in some copies
201	11	$\frac{a^n+b^n}{a-b}$	$\frac{a^n-b^n}{a-b}$
206	16	$-2a^2c^2$	$-2a^2c^2+2c^4$
222 *Ex.* 6.		4, 7, 48	4, 4, 49
238	1	DB	DP
277	22	280	284
292	21	KO : KO	KO : KL
309	6	12 *sec.*	6 *sec.*
328	21	$\frac{1}{2}dt=m$	$\frac{dt^2}{4s^2}=m$
		$h=\frac{1}{2}dt^2s^2$	$h=\frac{1}{4}dt^2$
	24	$s^2m=\frac{1}{2}dt^2s^2$	$s^2m=\frac{1}{4}dt^2$
	25	$\frac{h}{s^2}=\frac{1}{2}dt^2$	$\frac{h}{s^2}=\frac{dt^2}{4s^2}$
	26	$t=\sqrt{\frac{2h}{ds^2}}=\sqrt{\frac{2m}{d}}$	$t=2\sqrt{\frac{h}{d}}=2s\sqrt{\frac{m}{d}}$

336. *Ex.* 8. Let $T =$ *tang.* elevation; then $t=\sqrt{\frac{Ti}{d}}=11{\cdot}8$ *sec.* time of flight.

371. This solution (which is restricted to the case where the diam. of the base and height are equal, or when $b=h$) was transcribed through mistake instead of the following: Let VR $=b$, QS $=h$, $\cdot7854=m$, as before, and put SP $=a$, SO $=c$, &c. then CD $=\frac{ab}{h}$, AB $=\frac{cb}{h}$, &c. and $(h^3+a^3+c^3+$ &c. *in infin.*$)\times\frac{mb^2}{h^2}=\frac{h^4}{4}\times\frac{mb^2}{h^2}=\frac{mb^2h^2}{4}=$ all the *ds*, which divided by $\frac{mb^2h}{3}$ (the solid content) gives $\frac{3}{4}h$ the dist. of the centre of gravity from S.

Questions p. 472 &c.	*Corrected Answers.*
9	*Ans*: $\left(\frac{p^2}{b} - \frac{1}{4}b\right) \times$ *tang*. given angle.
31	*Ans.* $\begin{cases} \sqrt{\dfrac{4a^2 b^4 c^4}{4a^2 c^2 b^4 - (a^2 b^2 + b^2 c^2 - a^2 c^2)^2}} \\ \sqrt{\dfrac{4b^2 a^4 c^4}{4a^2 b^2 c^4 - (a^2 c^2 + b^2 c^2 - a^2 b^2)^2}}, \\ \sqrt{\dfrac{4c^2 a^4 b^4}{4b^2 c^2 a^4 - (a^2 b^2 + a^2 c^2 - b^2 c^2)^2}} \end{cases}$
41	*for* 7° 47′ *r.* 4° 47′ in the Quest.
44	*Ans.* 64.29 and 76.59, nearly.
52	*Ans.* $72\frac{3}{8} \times 4\frac{1}{2}$ *feet.*
53	*Ans.* 15544 *feet.*
66	*Ans.* 71 *inches*, nearly.
68	*for* 760 *r.* 766.
	38° 18′$\frac{1}{4}$ *r.* 35° 12′$\frac{1}{4}$
	43° 45′$\frac{2}{3}$ *r.* 43° 50′$\frac{1}{2}$
75	*for* 50 *yds.* *r.* 455 *yds.* in the Quest.
78	*Ans.* 33° 59′$\frac{1}{4}$
79	*Ans.* 8° 22′$\frac{3}{4}$
84	*Ans.* 25.97 *inch.* per second.
113	*Ans.* 3.11 casks, nearly.
125	*for* 50*lb.* *r.* 40*lb.* in the Quest.
131	*Ans.* 3778 *feet.*
132	*for* $2\frac{1}{4}$ *feet* *r.* $1\frac{1}{2}$ in the Quest.

TABLES

OF THE

LOGARITHMS

OF

NUMBERS,

FROM

1 TO 10000;

TOGETHER WITH THE

SINES AND TANGENTS,

TO

EVERY MINUTE OF THE QUADRANT.

LONDON:

Printed by W. GLENDINNING, 25, *Hatton Garden.*

1802.

LOGARITHMS

OF THE

NUMBERS,

FROM

1 to 10000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806180	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1 851258	96	1·982271
22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

N.	0	1	2	3	4	5	6	7	8	9
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174
102	8600	9026	9451	9876	0300	0724	1147	1570	1993	2415
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616
104	7033	7451	7868	8284	8700	9116	9532	9947	0361	0775
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
107	9384	9789	0195	0600	1004	1408	1812	2216	2619	3021
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028
109	7426	7825	8223	8620	9017	9414	9811	0207	0602	0998
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
112	9218	9606	9993	0380	0766	1153	1538	1924	2309	2694
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	0320
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
117	8186	8557	8928	9298	9668	0038	0407	0776	1145	1514
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	9181	9543	9904	0266	0626	0987	1347	1707	2067	2426
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
123	9905	0258	0611	0963	1315	1667	2018	2370	2721	3071
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	0026
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	0253
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	0245
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	0012
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
138	9879	0194	0508	0822	1136	1450	1763	2076	2389	2702
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911
141	9219	9527	9835	0142	0449	0756	1063	1370	1676	1982
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
144	8362	8664	8965	9266	9567	9868	0168	0469	0769	1068
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689
151	8977	9264	9552	9839	0126	0413	0699	0986	1272	1558
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	0051
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382
158	8657	8932	9206	9481	9755	0029	0303	0577	0850	1124
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247
162	9515	9783	0051	0319	0586	0853	1121	1388	1654	1921
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	0193
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795
173	8046	8297	8548	8799	9049	9299	9550	9800	0050	0300
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	0176
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937
185	7172	7406	7641	7875	8110	8344	8580	8812	9046	9279
186	9513	9746	9980	0213	0446	0679	0912	1144	1377	1609
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525
190	8754	8982	9211	9439	9667	9895	0123	0351	0578	0806
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635
199	8853	9071	9289	9507	9725	9943	0161	0378	0595	0813
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	298
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	513
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	728
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	941
204	9630	9843	0056	0268	0481	0693	0906	1118	1330	154
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	365
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	576
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	785
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	993
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	201
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	407
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	613
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	817
213	8380	8583	8787	8991	9194	9398	9601	9805	0008	021
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	223
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	425
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	626
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	825
218	8456	8656	8855	9054	9253	9451	9650	9849	0047	024
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	222
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	419
221	4392	4589	1785	4981	5178	5374	5570	5766	5962	615
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	811
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	0054
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	198
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	391
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646
229	9835	0025	0215	0404	0593	0783	0972	1161	1350	1539
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030
234	9216	9401	9587	9772	9958	0143	0328	0513	0698	0883
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	0030
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989
245	9166	9343	9520	9698	9875	0051	0228	0405	0582	0759
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
250	397940	8114	8287	8461	8634	8808	8981	9154	9328	9501
251	9674	9847	0020	0192	0365	0538	0711	0883	1056	1228
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764
257	9933	0102	0271	0440	0609	0777	0946	1114	1283	1451
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806
260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791
263	9956	0121	0286	0451	0616	0781	0945	1110	1275	1439
264	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591
269	9752	9914	0075	0236	0398	0559	0720	0881	1042	1203
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004
273	6163	6322	6481	6640	6800	6957	7116	7275	7433	7592
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175
275	9333	9491	9648	9806	9964	0122	0279	0437	0594	0752
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003
280	7158	7313	7468	7623	7778	7933	8088	8242	8397	8552
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	0095
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242
288	9392	9543	9694	9845	9995	0146	0296	0447	0597	0748
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675
295	9822	9969	0116	0263	0410	0557	0704	0851	0998	1145
296	1292	1438	1585	1732	1878	2025	2171	2318	2464	2610
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
300	477121	7266	7411	7555	7700	7844	7989	8133	8278	8422
301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863
302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818
309	9958	0099	0239	0380	0520	0661	0801	0941	1081	1222
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550
316	9687	9824	9962	0099	0236	0374	0511	0648	0785	0922
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068
323	9203	9337	9471	9606	9740	9874	0009	0143	0277	0411
324	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697
331	9828	9959	0090	0221	0353	0484	0615	0745	0876	1007
332	521138	1269	1400	1530	1661	1792	1922	2053	2183	2314
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788
338	8917	9045	9174	9302	9430	9559	9687	9815	9943	0072
339	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351
340	1479	1607	1734	1862	1990	2117	2245	2372	2500	2627
341	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899
342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432
344	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693
345	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951
346	9076	9202	9327	9452	9578	9703	9829	9954	0079	0204
347	540329	0455	0580	0705	0830	0955	1080	1205	133[illegible]	[illegible]454
348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701
349	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
350	544068	4192	4316	4440	4564	4688	4812	4936	5060	5183
351	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419
352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881
354	9003	9126	9249	9371	9494	9616	9739	9861	9984	0106
355	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547
357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182
360	6303	6423	6544	6664	6785	6905	7026	7146	7267	7387
361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589
362	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787
363	9907	0026	0146	0265	0385	0504	0624	0743	0863	0982
364	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084
370	8202	8319	8436	8554	8671	8788	8905	9023	9140	9257
371	9374	9491	9608	9725	9842	9959	0076	0193	0309	0426
372	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525
379	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669
380	9784	9898	0012	0126	0241	0355	0469	0583	0697	0811
381	580925	1039	1153	1267	1381	1495	1608	1722	1836	1950
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720
388	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838
389	9950	0061	0173	0284	0396	0507	0619	0730	0842	0953
390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066
391	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175
392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681
397	[illegible]91	8900	9009	9119	9228	9337	9446	9556	9665	9774
398	[illegible]83	9992	0101	0210	0319	0428	0537	0646	0755	0864
399	600973	1082	1191	1299	1408	1517	1625	1734	1843	1951
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
400	602060	2169	2277	2386	2494	2603	2711	2819	2928	3036
401	3144	3253	3361	3469	3373	3686	3794	3902	4010	4118
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488
407	9594	9701	9808	9914	0021	0128	0234	0341	0447	0554
408	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845
413	5950	6055	6160	6265	6370	6176	6581	6686	6790	6895
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	0032
417	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308
426	9410	9512	9613	9715	9817	9919	0021	0123	0224	0326
427	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383
432	5484	5581	5685	5785	5886	5986	6087	6187	6287	6388
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389
435	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387
436	9486	9586	9686	9785	9885	9984	0084	0183	0283	0382
437	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237
446	9335	9432	9530	9627	9724	9821	9919	0016	[illegible]	0210
447	650308	0405	0502	0599	0696	0793	0890	0987	[illegible]	1181
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
450	653213	3309	3405	3502	3598	3695	3791	3888	3984	4080
451	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042
452	5138	5235	5331	5427	5526	5619	5715	5810	5906	6002
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870
456	8965	9060	9155	9250	9346	9141	9536	9631	9726	9821
457	9916	0011	0106	0201	0296	0391	0486	0581	0676	0771
458	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718
459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663
460	2758	2852	2947	3041	3135	3230	3324	3418	3512	3607
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224
467	9317	9410	9503	9596	9689	9782	9875	9967	0060	0153
468	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080
469	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005
470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850
472	3942	4034	4126	4218	4310	4402	1494	4586	4677	4769
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602
175	6694	6785	6876	6968	7059	7151	7242	7333	7124	7516
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337
478	9428	9519	9610	9700	9791	9882	9973	0063	0154	0245
479	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151
480	1241	1332	1422	1513	1603	1693	1784	1874	1964	2055
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220
489	9309	9398	9486	9575	9664	9753	9841	9930	0019	0107
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
500	698970	9057	9144	9231	9317	9404	9491	9578	9664	9751
501	9838	9924	0011	0098	0184	0271	0358	0444	0531	0617
502	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	0033
513	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566
516	2650	2734	2818	2902	2986	3070	3154	3238	3326	3407
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	0077
525	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552
528	2634	2716	2798	2681	2963	3045	3127	3209	3291	3374
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893
537	9974	0055	0136	0217	0298	0378	0459	0540	0621	0702
538	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313
540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493
549	9572	9651	9731	9810	9889	9968	0047	0126	0205	0284
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
550	740363	0442	0521	0560	0678	0757	0836	0915	0994	1073
551	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860
552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2646
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777
557	5855	5933	6011	6080	6167	6245	6323	6401	6479	6556
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110
560	8188	8266	8343	8421	8498	8576	8653	8731	8808	8885
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659
562	9736	9814	9891	9968	0045	0123	0200	0277	0354	0431
563	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592
575	9668	9743	9819	9894	9970	0045	0121	0196	0272	0347
576	760422	0498	0573	0649	0724	0799	0875	0950	1025	1101
577	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853
578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604
579	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353
580	3428	3503	3578	3653	3727	3802	3877	3952	4027	4101
581	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848
582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303
588	9377	9451	9525	9599	9673	9746	9820	9894	9968	0042
589	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354
599	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
600	778151	8224	8296	8368	8441	8513	8585	8658	8730	8802
601	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524
602	9596	9669	9741	9813	9885	9957	0029	0101	0173	0245
603	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401
606	2473	2544	2616	2688	2750	2831	2902	2974	3046	3117
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832
608	3904	3975	4046	4118	4189	4261	4332	4403	4175	4546
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	9098
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510
616	9581	9651	9722	9792	9863	9933	0004	0074	0144	0215
617	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322
620	2392	2462	2532	2602	2672	2742	2812	2882	2952	3022
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115
624	5185	4254	5324	5393	5463	5532	5602	5672	5741	5811
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272
630	9341	9409	9478	9547	9616	9685	9754	9823	9892	9961
631	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112
640	6180	6248	6316	6384	6451	6519	6587	6655	6723	6790
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492
645	9560	9627	9694	9762	9829	9896	9964	0031	0098	0165
646	810233	0300	0367	0434	0501	0569	0636	0703	0770	0837
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508
648	1575	0642	1709	1776	1843	1910	1977	2044	2111	2178
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
650	812913	2980	3047	3114	3181	3247	3314	3381	3448	3514
651	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181
652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478
660	9544	9610	9676	9741	9807	9873	9939	0004	0070	0136
661	820201	0267	0333	0399	0464	0530	0595	0661	0727	0792
662	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448
663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103
664	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010
670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239
675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882
676	9947	0011	0075	0139	0204	0268	0332	0396	0460	0525
677	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445
680	2509	2573	2637	2700	2764	2828	2892	2956	3020	3083
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786
690	8849	8912	8975	9038	9101	9164	9227	9289	9352	9415
691	9478	9541	9604	9667	9729	9792	9855	9918	9981	0043
692	840106	0169	0232	1294	0357	0420	0482	0545	0608	0671
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415
699	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
700	845098	5160	5222	5284	5346	5408	5470	5532	5594	5656
701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972
708	850033	0095	0156	0217	0279	0340	0401	0462	0524	0585
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197
710	1258	1320	1381	1442	1503	1564	1625	1686	1747	1809
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272
720	7332	7393	7453	7513	7574	7634	7694	7755	7815	7875
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679
724	9739	9799	9859	9918	9978	0038	0098	0158	0218	0278
725	860338	0398	0458	0518	0578	0637	0697	0757	0817	0877
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263
730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173
740	9232	9290	9349	9408	9466	9525	9584	9642	9701	9760
741	9818	9877	9935	9994	0053	0111	0170	0228	0287	0345
742	870404	0462	0521	0579	0638	0696	0755	0813	0872	0930
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
750	875061	5119	5177	5235	5293	5351	5409	5466	5524	5582
751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160
752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612
758	9669	9726	9784	9841	9898	9956	0013	0070	0127	0185
759	880242	0299	0356	0413	0471	0528	0585	0642	0699	0756
760	0814	0871	0928	0985	1042	1099	1156	1213	1271	1328
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605
765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434
770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998
771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806
776	9862	9918	9974	0030	0086	0141	0197	0253	0309	0365
777	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039
780	2095	2150	2206	2262	2317	2373	2429	2484	2540	2595
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572
790	7627	7682	7737	7792	7847	7902	7957	8012	8067	8122
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766
794	9821	9875	9930	9985	0039	0094	0149	0203	0258	0312
795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
800	903090	3144	3199	3253	3307	3361	3416	3470	3524	3578
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431
810	8485	8539	8592	8616	8699	8753	8807	8860	8914	8967
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	0037
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169
817	2222	2275	2323	2381	2435	2488	2541	2594	2647	2700
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761
820	3814	3867	3920	3973	4026	4079	4132	4184	4237	4290
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026
830	9078	9130	9183	9235	9287	9340	9392	9444	9496	9549
831	9601	9653	9706	9758	9810	9862	9914	9967	0019	0071
832	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228
840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291
844	6342	6394	6415	6497	6548	6600	6651	6702	6754	6805
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
850	929419	9470	9521	9572	9623	9674	9725	9776	9827	9879
851	9930	9981	0032	0083	0134	0185	0236	0287	0338	0389
852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448
860	4498	4549	4599	4650	4700	4751	4801	4852	4902	4953
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470
868	8520	8570	8620	8670	8720	8770	8820	8870	8919	8970
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469
870	9519	9569	9619	9669	9719	9769	9819	9869	9918	9968
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341
890	9390	9439	9488	9536	9585	9634	9683	9731	9780	9829
891	9878	9926	9975	0024	0073	0121	0170	0219	0267	0316
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
900	954243	4291	4339	4387	4435	4484	4532	4580	4628	4677
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947
912	9995	0042	0090	0138	0185	0233	0280	0328	0376	0423
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835
933	9882	9928	9975	0021	0068	0114	0161	0207	0254	0300
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279
969	6321	6369	6413	6458	6503	6548	6593	6637	6682	6727
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514
974	8559	8601	8648	8693	8737	8782	8826	8871	8916	8960
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850
977	9895	9939	9983	0028	0072	0117	0161	0206	0250	0294
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392
985	3436	3480	3524	3568	3612	3657	3701	3745	3789	3833
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152
989	5196	5240	5284	5328	5372	5416	5460	5501	5547	5591
990	5635	5679	5723	5767	5811	5854	5898	5942	5986	6030
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957
N.	0	1	2	3	4	5	6	7	8	9